CLARK UNIVERSITY 1887

Conservative vector fields
Math 131 Multivariate Calculus
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Conservative vector fields. Recall that a *gradient field* F is the gradient ∇f of some vector field f, which is called a *potential field* for F. We're interested in what properties are required of a vector field F for it to be a gradient field. We'll use the term *conservative vector field* to mean the same thing as gradient field, but without a particular scalar field of which it's the gradient. We'll see that there are a couple of other equivalent conditions for a vector field to be a conservative field.

Path-independent vector line integrals. We'll start out by looking at vector fields with path-independent vector line integrals. We'll see soon that these fields are the gradient fields. In the following definition, we assume that the domain D of $\mathbf{F}$ is a connected region of $\mathbf{R}^n$, otherwise there won't be paths between any two points $\mathbf{a}$ and $\mathbf{b}$ of D.

Definition 1. We say a vector field $\mathbf{F}$ *has path-independent line integrals* in a connected domain D when each vector line integral $\displaystyle\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ depends only on the endpoints of the path $\mathbf{x}$. That is, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two paths with the initial endpoint $\mathbf{a}$ and same terminal endpoint $\mathbf{b}$, then

$$\int_{\mathbf{x}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}_2} \mathbf{F} \cdot d\mathbf{s}.$$

The unstated important part of this definition is that $\mathbf{x}_1$ and $\mathbf{x}_2$ don't even have to travel the same curve.

Theorem 2. A vector field $\mathbf{F}$ has path-independent line integrals if and only if all its integrals over simple closed curves C are zero:

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0.$$

Proof. One direction is easy. Suppose that $\mathbf{F}$ has path-independent line integrals. Let C be a simple closed curve, starting and ending at the same point $\mathbf{a}$. Another path with the same endpoint is the constant path, which we can denote $\mathbf{a}$. Since C and $\mathbf{a}$ have the same endpoints (both being $\mathbf{a}$), their integrals are the same.

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{a}} \mathbf{F} \cdot d\mathbf{s}.$$

But the path integral over a constant path is 0. Therefore, the integral over the closed curve C is also 0.

Now, let's look at the converse. Suppose that all its integrals of $\mathbf{F}$ over simple closed curves are zero. We need to show that $\mathbf{F}$ has path-independent line integrals. Let two paths $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same endpoints. Reorient $\mathbf{x}_2$ in the opposite direction and attach it to $\mathbf{x}_1$ to get a closed curve C. Then

$$\int_{\mathbf{x}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathbf{x}_2} \mathbf{F} \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{s}.$$

If C is a simple closed curve, then that last integral is 0, so the integrals over the paths $\mathbf{x}_1$ and $\mathbf{x}_2$ are equal. What's left to do for this proof is to show that if $\mathbf{F}$ has zero integrals over simple closed curves, then $\mathbf{F}$ has zero integrals over all closed curves. We can do the case where the closed curve C has only finitely many self intersections, but when there are infinitely many self intersections, the argument is more difficult. Q.E.D.

The next two theorems show that gradient fields are those fields having path-independent line integrals

Theorem 3. Gradient fields have path-independent line integrals.

Proof. Let $\mathbf{F} = \nabla f$ be a gradient field with potential field f. We'll show $\mathbf{F}$ has path-independent line integrals. Let $\mathbf{x}$ be a path in the domain D of $\mathbf{F}$. The argument depends on the multivariate chain rule and on the ordinary fundamental theorem of calculus.

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} \\ &= \int_a^b \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{x}(t))\; dt \\ &= f(\mathbf{x}(t))\Big|_a^b = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).\end{aligned}$$

Thus, the integral of $\mathbf{F}$ only depends on the the values of f at the endpoints $\mathbf{x}(a)$ and $\mathbf{x}(b)$, not on the path between the endpoints. Q.E.D.

Theorem 4. If a vector field has path-independent line integrals, then it's a gradient field.

Proof. As mentioned in the definition of path-independence, we assume that the domain D of the vector field $\mathbf{F}$ is connected, and here that means *path-connected*, that is, any two points in D can be joined by a path entirely in D.

We need to define a scalar-valued function f with domain D. Fix a point $\mathbf{a}$ in the domain D. For a point $b \in D$, define $f(\mathbf{b})$ as

$$f(\mathbf{b}) = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{x}$ is any path joining $\mathbf{a}$ to $\mathbf{b}$. Since $\mathbf{F}$ has path-independent integrals, this value $f(\mathbf{b})$ is independent of the path $\mathbf{x}$, so $f(\mathbf{b})$ is well-defined.

We won't do it, but you can show that $F = \nabla f$ by directly evaluating ∇f. That involves computing the i^{th} partial derivatives of f using the definition of derivative, and that comes down to finding short paths in the domain of $\mathbf{F}$ that are parallel to the ith axis and letting their lengths approach 0. Q.E.D.

Curls and gradient fields. Back when we first defined curl, we showed that the curl of a gradient field was 0, that is to say, gradient fields are irrotational. We did this by verifying the equation

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Now we'll show the converse, at least when the domain D is simply-connected plane region. First, we need to define what it means for a subset D of $\mathbf{R}^n$ to be simply connected. Intuitively, in $\mathbf{R}^2$ a set is simply connected if it has no holes. In $\mathbf{R}^3$ it's not enough to have no holes, since the unit sphere has a "hole" inside it, but the sphere turns out to be simply connected.

Definition 5. A connected set D in $\mathbf{R}^n$ is said to be *simply connected* if every simple closed curve C in D can be continuously shrunk to a point while remaining in R throughout the deformation. More precisely, there is a continuous function

$$\gamma : [0,1] \times [0,1] \to D$$

such that $\gamma(0)$ is a path $[0,1] \to \mathbf{R}^n$ that describes the curve C, and $\gamma(1)$ is a constant path.

In the plane, the interior D of a simply closed curve C is simply connected.

Theorem 6. If $\mathbf{F}$ is an irrotational vector field defined on a simply connected domain in $\mathbf{R}^2$, then $\mathbf{F}$ is a gradient field.

Proof. Let C be a simple closed curve in the region R. We'll show the path integral $\oint_c \mathbf{F} \cdot d\mathbf{s}$ is 0. That will imply $\mathbf{F}$ has path-independent line integrals, which is equivalent to being a gradient field.

Now, Green's theorem in the plane is equivalent to

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

In our case, D is the interior of the closed curve C. But the curl, $\nabla \times \mathbf{F}$, is 0, so the double integral is 0. Therefore, the line integral is 0. Q.E.D.

After we've proved Stokes' theorem, we'll be able to prove the analogous statement for dimension 3.

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