Critical points
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Critical points. If $\mathbf{F}$ is a gradient field in the plane $\mathbf{R}^2$ for the potential field f, then the flow lines for $\mathbf{F} = \nabla f$ are orthogonal to the equipotential curves.

We'll look at two important examples which show what can happen near critical points of f. A critical point is where $\nabla f = 0$. The tangent plane to $z = f(x,y)$ is horizontal at a critical point. Often a critical point will be a maximum or a minimum of f, but sometimes it will be a saddle point, and rarely it will be something more exotic. We'll look at the case that the critical point is a maximum (analogous phenomena occur when it's a minimum) and the case where it's a saddle point. The two examples are generic in the sense that they show what's going on in the general case.

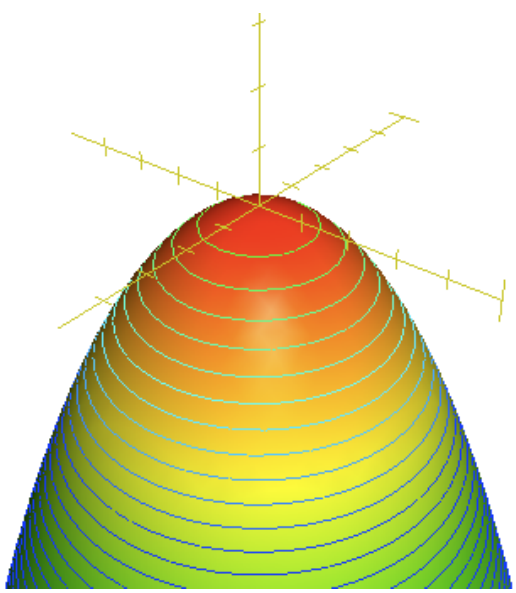

Figure 1: Maximum

Example 1 (A maximum)**.** Let the potential function be
$$f(x) = -x^2 - y^2$$
which is illustrated in figure 1.

It clearly has a maximum at $(0,0)$. The equipotential curves are concentric circles $x^2 + y^2 = C$ with center at $(0,0)$. The gradient field is $\mathbf{F}(x,y) = \nabla f(x,y) = (-2x, -2y)$. A path $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ is a flow line if
$$\begin{cases} x'(t) &= -2x(t) \\ y'(t) &= -2y(t) \end{cases}$$
These exponential differential equations have the solutions
$$\begin{cases} x(t) &= Ae^{-2t} \\ y(t) &= Be^{-2t} \end{cases}$$
where the constants give the initial position $\mathbf{x}(0) = (x(0), y(0)) = (A, B)$. Note that as $t \to \infty$, the point $\mathbf{x}$ approaches $(0,0)$, but exponentially slowly. We can eliminate t from the pair of equations defining x and y to see what curve the path lies on. We get $x/y = A/B$. That's a straight line through $(0,0)$ with slope. Thus, the flows move along straight lines headed towards $(0,0)$ slowing down exponentially.

If we change the potential function to $f(x) = x^2+y^2$, we get an example with a minimum at $(0,0)$. The flows move along the same straight lines, but headed away from $(0,0)$ speeding up exponentially.

Example 2 (A saddle point)**.** A small modification to the preceding example gives an example of a saddle point. Let the potential function be
$$f(x) = x^2 - y^2$$
illustrated in figure 2.

The gradient field is $\mathbf{F}(x,y) = \nabla f(x,y) = (2x, -2y)$. The only critical point occurs at $(0,0)$. It's not a minimum, since smaller values of f occur on the y-axis, and it's not a maximum, since larger values occur on the x-axis. It's a saddle point.

The equipotential curves have equations $x^2 - y^2 = C$. These are hyperbolas with asymptotes $y = \pm x$.

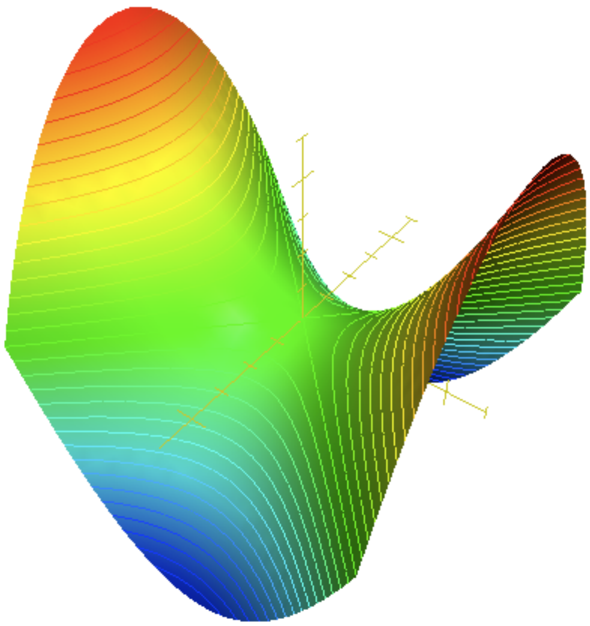

Figure 2: A saddle point

A path $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ is a flow line if

$$\begin{cases} x'(t) &=& 2x(t) \\ y'(t) &=& -2y(t) \end{cases}$$

These exponential differential equations have the solutions

$$\begin{cases} x(t) &=& Ae^{2t} \\ y(t) &=& Be^{-2t} \end{cases}$$

where, again, the constants give the initial position $\mathbf{x}(0) = (x(0), y(0)) = (A, B)$. Along the y-axis we have $A = 0$ and $(x(0), y(t)) = (0, Be^{2t})$, so the flow approaches the origin slowing down exponentially as it approaches the origin. On the other hand, along the x-axis we have $B = 0$ and $(x(t), y(0)) = (Ae^{2t}, 0)$, so the flow leaves the origin, speeding up exponentially as it leaves the origin.

We can eliminate t from the pair of equations defining x and y to see what curve the path lies on. We get $xy = AB$. That's a hyperbola with asymptotes being the x- and y-axes. Thus, the flows move along these hyperbolas, slow near the origin but fast away from the origin.

Later on, we'll study critical points in detail.

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