The unit tangent vector and curvature
Math 131 Multivariate Calculus

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Summary of the arclength pararamter s. The length of a path $\mathbf{x} : [a,b] \to \mathbf{R}^n$, also called its arclength, is the integral of its speed

$$\int_a^b \|\mathbf{x}'(t)\|\,dt = \int_a^b \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\,dt.$$

Denote the length of the path up to t as $s(t)$, that is,

$$s(t) = \int_a^t \|\mathbf{x}'\| = \int_a^t \|\mathbf{x}'(\tau)\|\,d\tau.$$

Then by the fundamental theorem of calculus,

$$\frac{ds}{dt} = \|\mathbf{x}'(t)\| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2},$$

which says that the derivative of the arclength s is the speed. In other words, ds is the differential of the arclength s. The last equation can be written in a differential form as

$$ds = \sqrt{dx_1^2 + \cdots + dx_n^2}.$$

The unit tangent vector and arclength. The velocity vector, $\mathbf{v}(t) = \mathbf{x}'(t)$, for a path $\mathbf{x}$, points in a direction tangent to the path at the point $\mathbf{x}(t)$. We can normalize it to make it a unit tangent vector $\mathbf{T}$ just by dividing it by its length:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{x}'}{\|\mathbf{x}'\|}.$$

Of course, this is only defined when $\mathbf{x}'(t)$ is not $\mathbf{0}$. Note that T could also be defined as

$$\mathbf{T} = \frac{\mathbf{dx}}{ds}$$

since we saw earlier that $\dfrac{d\mathbf{x}}{ds}$ was equal to $\frac{\mathbf{x}'}{\|\mathbf{x}'\|}$.

The unit tangent vector $\mathbf{T}$ gives the direction of the curve. It's useful because it says which way the path is going, but doesn't indicate how fast the object is travelling that path. Thus, $\mathbf{T}$ is an intrinsic property of the underlying curve.

We can use $\mathbf{T}$ to study how the curve bends, since the bend of the curve has to do with the change in the direction of the curve. Since $\mathbf{T}$ is a unit vector, we can identify it with an angle between 0 and 2π. The rate of change of $\mathbf{T}$, therefore, has to do with the rate of change of this angle, in fact, it is the derivative of that angle.

Theorem 1. Let $\mathbf{x}$ be a path with nonzero speed. Then

1. $\dfrac{d\mathbf{T}}{dt}$ is orthogonal to $\mathbf{T}$.

2. $\|\dfrac{d\mathbf{T}}{dt}\|$ is the angular rate of change of the direction of $\mathbf{T}$, in other words $\|d\mathbf{T}\| = |d\theta|$, where $d\theta$ is the differential angle associated to the differential $d\mathbf{T}$.

We saw part 1 recently when we showed it was true for any unit vector $\mathbf{u} : \mathbf{R} \to \mathbf{R}^2$, but it's just as true when $n > 2$. The other statement is also true of any unit vector $\mathbf{u}$, and it depends on the differential angle $d\theta$ being equal to the $d\mathbf{u}$, a vector tangent to the unit circle (or unit sphere). A diagram helps here.

The curvature, or bend, of a curve is suppose to be the rate of change of the direction of the curve, so that's how we define it.

Definition 2 (curvature)**.** Let $\mathbf{x}$ be a path with unit tangent vector $\mathbf{T} = \dfrac{\mathbf{x}'}{\|\mathbf{x}'\|}$. The *curvature* κ at t is the angular rate of change of $\mathbf{T}$ per unit change in the distance along the path. That is,

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\|.$$

By the chain rule, this can also be written

$$\kappa = \frac{\|d\mathbf{T}/dt\|}{|ds/dt|}.$$

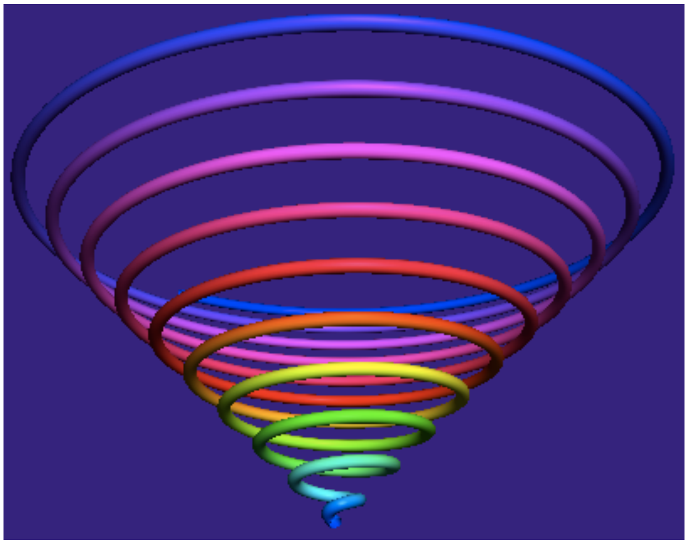

Figure 1: Conic helix

Curves that are nearly straight have nearly 0 curvature, while those that curl up tightly have high curvature.

Figure 1 shows a conic helix. Its equation in cylindrical coordinates is $(r, \theta, z) = (t, t, t)$, and in rectangular coordinates $(x, y, z) = (t \cos t, t \sin t, t)$. Near the apex of the cone, $(0, 0, 0)$, it's curved tight with a high curvature, but as it moves away from that radius of the spiral gets larger and the curvature decreases.

It's easy to show that a circle of radius r has curvature $\kappa = 1/r$. In fact, an alternate definition for curvature is that it is the reciprocal of the radius of the circle that best fits the curve at the point in question.

Example 3 (The helix again)**.** From the symmetry of a helix, you can expect the curvature to be the same at every point. First let's compute the unit tangent vector $\mathbf{T}$ for the helix.

Since

$$\mathbf{x}(s) = \left(a \cos \frac{s}{\sqrt{a^2+b^2}}, a \sin \frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}} \right),$$

therefore

$$\begin{aligned} T(s) &= \frac{d\mathbf{x}}{ds} \\ &= \left(\frac{-a}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}}, \right. \\ &\qquad \left. \frac{a}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right) \end{aligned}$$

Now differentiate this direction $\mathbf{T}$ with respect to s to get

$$\frac{dT}{ds} = \left(\frac{-a}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}}, \frac{-a}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}}, 0 \right)$$

Then, leaving out the algebra, the curvature is

$$\kappa = \left\| \frac{dT}{ds} \right\| = \frac{a}{a^2+b^2}.$$

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