Name: ____________________________

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Math 131 Multivariate Calculus

Sample Final Exam

You may refer to one sheet of notes on this test. Points for each problem are in square brackets.

Problem 1. [12] On Gauss's theorem. Consider the following surface integral over the unit sphere $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.

$$\iint_S (x - y)\, dx + (y - z)\, dy + (z - x)\, dz.$$

a. Use Gauss's theorem to convert this integral to an ordinary triple integral. You may leave the triple integral over a region D, but describe what the region D is.

b. Explain why this particular triple integral is easy to evaluate, then state its value.

Problem 2. [12] On surfaces. Parametrize the torus (a surface that looks like the surface of a doughnut) by

$$\mathbf{X}(s, t) = ((5 + 2\cos t)\cos s, (5 + 2\cos t)\sin s, 2\sin t)$$

where both s and t range from 0 through 2π.

a. Describe $\mathbf{X}(s, 0)$, the s-coordinate curve (latitude) at $t = 0$.

b. Evaluate the tangent vector $\mathbf{T}_s(\pi/2, 0)$ to that s-coordinate curve $\mathbf{X}(s, 0)$ at $s = \pi/2$.

c. Verify that the normal vector $\mathbf{N}(s, t)$ equals $(10 + 4\cos t)(\cos s \cos t\, \mathbf{i} - \sin s \cos t\, \mathbf{j} + \sin t\, \mathbf{k})$.

d. Write down a surface integral over S that gives the area of this torus, and convert it into an ordinary double integral. Your double integral should only involve the variables s and t, and it should have specific limits of integration. You may use the fact that $\|\mathbf{N}(s, t)\| = |10 + 4\cos t|$. Don't evaluate the integral.

Problem 3. [10] On change of variables. Consider the double integral

$$\iint_D (2x + y)^2 e^{x-y}\, dA$$

where D is the region enclosed by the lines $2x+y = 1$, $2x+y = 4$, $x-y = -1$, and $x-y = 1$.

a. Determine a substitution (u, v) in terms of (x, y) that will simplify this double integral, and evaluate the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ for this substitution.

b. Convert the integral into one in terms of the variables u and v. Your answer should be a double integral that involves only the variables u and v, and it should have specific limits of integration for u and v. Don't evaluate the resulting integral.

Problem 4. [12] On the Hessian and the second derivative test. Consider the function $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$.

a. Determine all the first and second derivatives of f.

b. Determine the critical point of f. (There is just one.)

c. Use the Hessian criterion to determine the nature of the critical point (max, min, or saddle).

Problem 5. [10] On scalar line integrals. Consider the line integral

$$\int_{\mathbf{x}} \frac{x+z}{y+z}\, ds$$

over the path $\mathbf{x}(t) = (t, t, t^{3/2})$ for $0 \le t \le 3$. Evaluate this integral to the point where you have an ordinary integral in terms of t; no other variable should appear in your integral.

Problem 6. [10] On multiple integrals. Evaluate the double integral

$$\int_1^2 \int_{-x}^{x} (y^2 + x)\, dy\, dx.$$

Problem 7. [12] Here are some true/false questions. If the statement is always true, then write 'true', but if it is sometimes false, or if it is meaningless, then write 'false'. (No explanation is required.) For each statement, assume the scalar or vector field mentioned in the statment is C^2.

a. The divergence of the curl of a vector field is zero.

b. The gradient of the divergence of a vector field is zero.

c. The curl of the gradient of a scalar field is zero.

d. The curl of the divergence of a scalar field is zero.

Problem 8. [12] On paths. Calculate the velocity, speed, acceleration, and unit tangent vector of the path $\mathbf{x}(t) = (t\cos t, t\sin t, t^2)$.

a. Velocity.

b. Speed.

c. Acceleration.

d. Unit tangent vector.

Problem 9. [10] On the chain rule. Suppose that vector field $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ has the derivative

$$D\mathbf{f}(x,y) = \begin{bmatrix} \cos z & 0 & -x\sin z \\ 0 & \sin z & y\cos z \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{x} : \mathbf{R} \to \mathbf{R}^3$ has the derivative $D\mathbf{x}(t) = \begin{bmatrix} e^t \\ -e^{-1} \\ 1 \end{bmatrix}$.

a. The derivative $D(\mathbf{f} \circ \mathbf{x})(t)$ is a matrix. What size is that matrix?

b. Find the derivative $D(\mathbf{f} \circ \mathbf{x})(t)$. You may leave your answer in terms of t, x, y, and z.

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#2.[12]	
#3.[10]	
#4.[12]	
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#9.[10]	
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