Math 131, Multivariate Calculus

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Sample Final Exam Answers

Problem 1. [12] On Gauss's theorem. Consider the following surface where $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$ is the unit sphere.

$$\iint_S (x-y)\,dx + (y-z)\,dy + (z-x)\,dz.$$

a. Use Gauss's theorem to convert this integral to an ordinary triple integral. You may leave the triple integral over a region D, but describe what the region D is.

The sphere S is the boundary ∂D of the solid ball

$$D = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}.$$

The given integral

$$\iint_{\partial D} (x-y, y-z, z-x)\cdot d\mathbf{S}$$

is the vector surface integral the integral of the vector field $\mathbf{F}(x,y,z) = (x-y, y-z, z-x)$. Gauss's theorem says that integral equals the integral of the divergence of $\mathbf{F}$ over the 3-dimensional region D

$$\iiint_D \nabla\cdot\mathbf{F}\,dV.$$

But $\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1+1+1 = 3$, so the integral simplifies to

$$\iiint_D 3\,dV.$$

b. Explain why this particular triple integral is easy to evaluate, then state its value.

It's just 3 times the volume of D. The volume of D is $\frac{4}{3}\pi$, so the integral equals 4π.

Problem 2. [12] On surfaces. Parametrize the torus (a surface that looks like the surface of a doughnut) by

$$\mathbf{X}(s,t) = ((5+2\cos t)\cos s, (5+2\cos t)\sin s, 2\sin t)$$

where both s and t range from 0 through 2π.

a. Describe $\mathbf{X}(s,0)$, the s-coordinate curve (latitude) at $t=0$.

The path $\mathbf{X}(s,0) = (7\cos x, 7\sin x, 0)$ describes a circle of radius 7 in the (x,y)-plane centered at the origin.

b. Evaluate the tangent vector $\mathbf{T}_s(\pi/2, 0)$ to that s-coordinate curve $\mathbf{X}(s,0)$ at $s=\pi/2$.

The tangent vector is the derivative of the s-coordinate curve.

$$\mathbf{T}_s(s,0) = \frac{\partial}{\partial s}\mathbf{X}(s,0) = (-7\sin s, 7\cos s, 0).$$

Evaluated at $(\pi/2, 0)$ that equals $\mathbf{T}_s(\pi/2,0) = (-7,0,0)$.

c. Verify that the normal vector $\mathbf{N}(s,t)$ equals $(10+4\cos t)(\cos s\cos t\,\mathbf{i} - \sin s\cos t\,\mathbf{j} + \sin t\,\mathbf{k})$.

$$\begin{aligned}
&\mathbf{N}(s,t)\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(5+2\cos t)\sin s & (5+2\cos t)\cos s & 0 \\ -2\sin t\cos s & -2\sin t\sin s & 2\cos t \end{vmatrix}\\
&= (10+4\cos t)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin s & \cos s & 0 \\ -\sin t\cos s & -\sin t\sin s & \cos t \end{vmatrix}\\
&= (10+4\cos t)(\cos s\cos t\,\mathbf{i} - \sin s\cos t\,\mathbf{j} + \sin t\,\mathbf{k})
\end{aligned}$$

d. Write down a surface integral over S that gives the area of this torus, and convert it into an ordinary double integral. Your double integral should only involve the variables s and t, and it should have specific limits of integration. You may use the fact that $\|\mathbf{N}(s,t)\| = |10+4\cos t|$. Don't evaluate the integral.

The area is the integral of 1 over the surface.

$$\iint_S 1\,d\mathbf{S} = \iint_S 1\,\|\mathbf{N}(s,t)\|\,ds\,dt = \int_0^{2\pi}\int_0^{2\pi} |10+4\cos t|\,ds\,dt$$

Problem 3. [10] On change of variables. Consider the double integral

$$\iint_D (2x+y)^2 e^{x-y}\,dA$$

where D is the region enclosed by the lines $2x+y=1$, $2x+y=4$, $x-y=-1$, and $x-y=1$.

a. Determine a substitution (u, v) in terms of (x, y) that will simplify this double integral, and evaluate the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ for this substitution.

A good substitution is $u = 2x + y$ and $v = x - y$. Then the inverse Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3.$$

The direct Jacobian is the reciprocal of that

$$\frac{\partial(x,y)}{\partial(u,v)} = -\tfrac{1}{3}.$$

b. Convert the integral into one in terms of the variables u and v. Your answer should be a double integral that involves only the variables u and v, and it should have specific limits of integration for u and v. Don't evaluate the resulting integral.

$$\iint_{D^*} u^2 e^v \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = \int_{-1}^{1} \int_{1}^{4} u^2 e^v \tfrac{1}{3}\, du\, dv.$$

Problem 4. [12] On the Hessian and the second derivative test. Consider the function $f(x, y) = 2xy - 2x^2 - 5y^2 + 4y - 3$.

a. Determine all the first and second derivatives of f.

$$f_x = 2y - 4x \qquad f_y = 2x - 10y + 4$$

$$f_{xx} = -4 \qquad f_{xy} = 2 \qquad f_{xy} = -10$$

b. Determine the critical point of f. (There is just one.)

The first partials are both 0 only at $\mathbf{a} = (\frac{2}{9}, \frac{4}{9})$. So $\mathbf{a}$ is the only critical point.

c. Use the Hessian criterion to determine the nature of the critical point (max, min, or saddle).

$$Hf(\mathbf{a}) = \begin{bmatrix} f_{xx}(\mathbf{a}) & f_{xy}(\mathbf{a}) \\ f_{yx}(\mathbf{a}) & f_{yy}(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -10 \end{bmatrix}$$

Now evaluate the two principal minors d_1 and d_2. The first principal minor d_1 is just -4, the upper left entry of the Hessian. The second principal minor d_2 is the determinant of the Hessian,

$$d_2 = |Hf(\mathbf{a})| = \begin{vmatrix} -4 & 2 \\ 2 & -10 \end{vmatrix} = 36.$$

Since all the odd principal minors (there's only one of them), are negative, while all the even principal minors (again, there's only one of them), are positive, therefore the second derivative test says this critical point is a local maximum.

Problem 5. [10] On scalar line integrals. Consider the line integral

$$\int_{\mathbf{x}} \frac{x+z}{y+z}\, ds$$

over the path $\mathbf{x}(t) = (t, t, t^{3/2})$ for $0 \le t \le 3$. Evaluate this integral to the point where you have an ordinary integral in terms of t; no other variable should appear in your integral.

$\mathbf{x}' = (1, 1, \frac{3}{2}t^{1/2})$. $\|\mathbf{x}'\| = \sqrt{1 + 1 + \frac{9}{4}t} = \frac{1}{2}\sqrt{8 + 9t}$.

$$\begin{aligned} \int_{\mathbf{x}} \frac{x+z}{y+z}\, ds &= \int_0^3 \frac{x+z}{y+z}\, \tfrac{1}{2}\sqrt{8+9t}\, dt \\ &= \tfrac{1}{2} \int_0^3 \frac{t + \frac{3}{2}t^{1/2}}{t + \frac{3}{2}t^{1/2}} \sqrt{8+9t}\, dt \\ &= \tfrac{1}{2} \int_0^3 \sqrt{8+9t}\, dt \end{aligned}$$

If you went on to evaluate this integral, you would get

$$\tfrac{1}{27}(8+9t)^{3/2}\Big|_0^3 = \tfrac{1}{27}35^{3/2}.$$

Problem 6. [10] On multiple integrals. Evaluate the double integral

$$\int_1^2 \int_{-x}^{x} (y^2 + x)\, dy\, dx.$$

First, evaluate the inner integral

$$\begin{aligned} \int_{-x}^{x} (y^2 + x)\, dy &= \tfrac{1}{3}y^3 + xy\Big|_{-x}^{x} \\ &= \tfrac{2}{3}x^3 + 2x^2 \end{aligned}$$

Next, put that back in the double integral and evaluate the outer integral.

$$\int_1^2 (\tfrac{2}{3}x^3 + 2x^2)\, dx = \tfrac{1}{6}x^4 + \tfrac{2}{3}x^3\Big|_1^2 = \tfrac{43}{6}$$

Problem 7. [12] Here are some true/false questions. If the statement is always true, then write 'true', but if it is sometimes false, or if it is meaningless, then write 'false'. (No explanation is required.) For each statement, assume the scalar or vector field mentioned in the statment is C^2.

a. The divergence of the curl of a vector field is zero.

It is true that $\nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}$.

b. The gradient of the divergence of a vector field is zero.

It is not always true that $\nabla(\nabla \cdot \mathbf{F}) = \mathbf{0}$. Not all vector fields have constant divergence.

c. The curl of the gradient of a scalar field is zero.

It is true that $\nabla \times (\nabla f) = \mathbf{0}$.

d. The curl of the divergence of a scalar field is zero.

This doesn't make sense. A scalar field doesn't have a divergence. Also, the divergence of a vector field is a scalar field, so that divergence can't have a curl.

Problem 8. [12] On paths. Calculate the velocity, speed, acceleration, and unit tangent vector of the path $\mathbf{x}(t) = (t\cos t, t\sin t, t^2)$.

a. Velocity.

$$\mathbf{x}'(t) = (\cos t - t\sin t, \sin t + t\cos t, 2t).$$

b. Speed.

$$\begin{aligned} \|\mathbf{x}'(t)\| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 4t^t} \\ &= \sqrt{1+5t^2} \end{aligned}$$

c. Acceleration.

$$\mathbf{x}''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t, 2).$$

d. Unit tangent vector.

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{(\cos t - t\sin t, \sin t + t\cos t, 2t)}{\sqrt{1+5t^2}}$$

Problem 9. [10] On the chain rule. Suppose that vector field $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ has the derivative

$$D\mathbf{F}(x,y,z) = \begin{bmatrix} \cos z & 0 & -x\sin z \\ 0 & \sin z & y\cos z \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{x} : \mathbf{R} \to \mathbf{R}^3$ has the derivative $D\mathbf{x}(t) = \begin{bmatrix} e^t \\ -e^{-1} \\ 1 \end{bmatrix}$.

a. The derivative $D(\mathbf{F} \circ \mathbf{x})(t)$ is a matrix. What size is that matrix?

Since $\mathbf{F} \circ \mathbf{x} : \mathbf{R} \to \mathbf{R}^3$, its derivative is a 3×1 matrix.

b. Find the derivative $D(\mathbf{F} \circ \mathbf{x})(t)$. You may leave your answer in terms of t, x, y, and z.

$$\begin{bmatrix} \cos z & 0 & -x\sin z \\ 0 & \sin z & y\cos z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t \\ -e^{-1} \\ 1 \end{bmatrix} = \begin{bmatrix} e^t\cos z - x\sin z \\ -e^{-1}\sin z + y\cos z \\ 1 \end{bmatrix}$$