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Green's theorem
Math 131 Multivariate Calculus

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Introduction. We'll introduce but not prove Green's theorem today. We'll see how it leads to what are called Stokes' theorem and the divergence theorem in the plane. Next time we'll outline a proof of Green's theorem, and later we'll look at Stokes' theorem and the divergence theorem in 3-space.

Green's theorem as a generalization of the fundamental theorem of calculus. Recall one form of the fundamental theorem of calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This theorem equates the integral of one function, namely $f'(x)$, over a 1-dimensional region $[a,b]$ to the difference of the values of a related function, namely $f(x)$, at the boundary of that region, the boundary being the endpoints of the interval.

Is there a 2-dimensional analogue? Can we find an equation that equates a double integral $\iint_D F(x,y)\,dx\,dy$ over a 2-dimensional region D to an integral $\int_C$ of some related function over the boundary C of D? Since D is a region in the plane, its boundary, $C = \partial D$, is a curve, or perhaps several curves if D has holes in it.

Yes, there is a 2-dimensional analogue, and it's called Green's theorem, or sometimes Ostrogradsky's theorem. Here it is.

Let D be a closed, bounded regtion in $\mathbf{R}^2$ with boundary $C = \partial D$ which is one or a finite number of closed curves. Let the closed curves of C be oriented so that D is on the left as C is traversed. Let $\mathbf{F} = (M,N)$ be a vector field, that is, M and N are both scalar fields. Then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Here, the symbol $\oint$ is just a variant of $\int$ that's often used for line integrals when the line is a closed curve or a finite union of closed curves.

Example 1. Let $\mathbf{F} = (M,N) = (2y,x)$ and D is the semicircular region $x^2+y^2 \le a^2$ with $y \ge 0$. The 2-dimensional region D includes the interior of the semicircle, while its boundary $C = \partial D$ is the closed curve only (made up of half the circumference of a circle and a line segment).

Green's theorem equates a path integral $\oint_{\partial D}$ over the boundary ∂D of a region D to a double integral $\iint_D$ over the region. If $\mathbf{F}$ is a plane vector field with coordinate functions M and N, $\mathbf{F} = (M,N)$, then Green's theorem says

$$\oint_{\partial D} \mathbf{F}\cdot ds = \oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Green's theorem can be interpreted in a couple of ways that give it some meaning, and later we'll generalize these interpretations as theorems in $\mathbf{R}^3$. The first interpretation is a version of *Stokes' theorem*. Stokes' theorem involves the curl of the vector field $\mathbf{F}$. The second interpretation is the divergence theorem (also called Gauss' theorem) in the plane which, of course, involves the divergence of the vector field $\mathbf{F}$.

Stokes' theorem for the plane. The vector field $\mathbf{F} = (M,N)$ is a two-dimensional vector field, not a three-dimensional one, so it doesn't have a curl as we defined curl. But two-dimensional vector fields are commonly assigned curls as follows. Make $\mathbf{F}$ to be a three-dimensional vector field by setting the third component function to be 0. Then $\mathbf{F} = (M,N,0)$ has a curl $\nabla \times \mathbf{F}$, and we can calculate that curl.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

This curl $\nabla \times \mathbf{F}$ is a vector, but it only has a component in the $\mathbf{k}$ direction. We can dot it with $\mathbf{k}$ to formally make it into a scalar value

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

since $\mathbf{k} \cdot \mathbf{k} = 1$. That means we can rewrite Green's theorem as

$$\oint_{\partial D} \mathbf{F} \cdot ds = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA.$$

In words, that says the integral of the vector field $\mathbf{F}$ around the boundary ∂D equals the integral of the curl of $\mathbf{F}$ over the region D.

The divergence theorem in the plane. For each pont on the curve ∂D, let $\mathbf{n}$ be the *outward unit normal vector*, that is, a unit vector orthogonal to the curve and pointing away from the region D. Then the divergence theorem says

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \nabla \cdot \mathbf{F} \, dA.$$

Proof. The boundary ∂D is made of one or more simple closed curves. Let $\mathbf{x}(t)$ for $a \le t \le b$ parameterize one of them. The unit tangent vector at a point $\mathbf{x}(t)$ on this curve is

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{(x'(t), y'(t))}{\|\mathbf{x}'(t)\|}$$

When you rotate the unit tangent vector $\mathbf{T}$ by 90° clockwise, you get the normal vector $\mathbf{n}$:

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\|\mathbf{x}'(t)\|}.$$

Therefore,

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{n} \, ds &= \int_a^b \mathbf{F} \cdot \mathbf{n}(t) \, \|\mathbf{x}'(t)\| \, dt \\ &= \int_a^b (M, N) \cdot \big(y'(t), -x'(t)\big) \, dt \\ &= \int_a^b \big(My'(t) - Nx'(t)\big) \, dt \\ &= \int_{\mathbf{x}} M \, dy - N \, dx \end{aligned}$$

Since that equation holds for every component curve in ∂D, it holds for the whole boundary:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{\partial D} -N \, dx + M \, dy.$$

But Green's theorem says

$$\int_{\partial D} -N \, dx + M \, dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial(-N)}{\partial y} \right) dA,$$

therefore

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA.$$

But that last integral is just $\displaystyle\iint_D \nabla \cdot \mathbf{F} \, dA$. Q.E.D.

In words, this divergence theorem says that the integral around the boundary ∂D of the the normal component of the vector field $\mathbf{F}$ equals the double integral over the region D of the divergence of $\mathbf{F}$.

When $\mathbf{F}$ is the velocity of a flow on the plane, then its normal component $\mathbf{F} \cdot \mathbf{n}$ gives the rate of flow at that point on the boundary, and the integral $\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds$ describes the total flow rate across ∂D, called the *flux* of $\mathbf{F}$ across ∂D.

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