CLARK UNIVERSITY 1887

Limits of functions of several variables

Math 131 Multivariate Calculus

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The definition of limits. We're going to define derivatives for multivariate functions in terms of limits just as we defined derivatives for ordinary functions in calculus. So, before we get to derivatives, we'll first have to define limits of multivariate functions.

Given a function $\mathbf{f}:\mathbf{R}^n\to\mathbf{R}^m$ we want to define the limit

$$\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})=\mathbf{L}.$$

Sometimes we'll also express that limit notationally as

$$\text{As } \mathbf{x}\to\mathbf{a}, \mathbf{x}\to\mathbf{L}.$$

These notations are meant to express the concept that as the vector $\mathbf{x}$ approaches the constant vector $\mathbf{a}$ in $\mathbf{R}^n$, the vector $\mathbf{f}(\mathbf{x})$ approaches the constant vector $\mathbf{L}$ in $\mathbf{R}^m$.

As in calculus, we'll use quantified ϵ's and δ's. Intuitively, the definition will say that $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})=\mathbf{L}$ if and only if the distance between $\mathbf{f}(\mathbf{x})$ and $\mathbf{L}$, that is, $\|\mathbf{f}(\mathbf{x})-\mathbf{L}\|$, can be made arbitrarily small by making the distance between $\mathbf{x}$ and $\mathbf{a}$, that is, $\|\mathbf{x}-\mathbf{a}\|$, sufficiently small.

More precisely, $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})=\mathbf{L}$ if and only if for each $\epsilon>0$ there is a $\delta>0$ such that $0<\|\mathbf{x}-\mathbf{a}\|<\delta$ implies $\|\mathbf{f}(\mathbf{x})-\mathbf{L}\|<\epsilon$.

Just as in ordinary calculus, we usually can't evaluate $\mathbf{f}$ at $\mathbf{a}$. In particular, we're going to define derivatives in terms of limits of quotients as $h\to 0$, and as each of those quotients will have h in the denominator, therefore the quotient won't be defined when $h=0$. That explains why we have the condition $0<\|\mathbf{x}-\mathbf{a}\|$ in the definition; it's just another way of saying $\mathbf{x}\neq\mathbf{a}$.

About some topological concepts. In the text there's a discussion of various topological concepts—*open*, *closed*, *boundary*, *neighborhood*, and *accumulation point*. These are particularly important when the concepts of limit and continuity are extended to more general spaces, but for the time being, we'll just survey them in passing. They won't be used for a while.

Properties of limits. Recall from calculus the important properties of limits of functions $f:\mathbf{R}\to\mathbf{R}$.

First, if a limit exists, then it is unique. That is, if $L\neq M$, then $\lim_{x\to a}f(x)$ can't be both L and M. That property justifies using the notation of equality in the expression $\lim_{x\to a}f(x)=L$. The analogous property holds for vector limits $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})$, and the reason is pretty much the same, but at one point in the proof, the triangle inequality is necessary for the vector case. The triangle inequality for scalars reads

$$|a-c|\leq|a-b|+|b-c|,$$

where a, b, and c are scalars. The triangle inequality for vectors $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{R}^n$ says

$$\|\mathbf{a}-\mathbf{c}\|\leq\|\mathbf{a}-\mathbf{b}\|+\|\mathbf{b}-\mathbf{c}\|,$$

and for vectors, there really is a triangle. When $\mathbf{b}$ is the $\mathbf{0}$ vector, you get a simpler form of the triangle inequality, namely,

$$\|\mathbf{a}-\mathbf{c}\|\leq\|\mathbf{a}\|+\|\mathbf{c}\|.$$

Next, for functions of one variable, the limit of a constant is that constant, that is

$$\lim_{x\to a}c=c$$

where c is any scalar constant. The same thing holds for vectors, that is,

$$\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{c}=\mathbf{c}$$

where $\mathbf{c}$ is any constant vector in $\mathbf{R}^n$.

Also, the limit of x as $x \to a$ is a itself, that is,

$$\lim_{x\to a} x = a.$$

Of course, that will be true in the vector case, too:

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{x} = \mathbf{a}.$$

Next for functions of one variable, there's the limit of the sum is the sum of the limit, that is,

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Again, that property holds too for multivariate limits

$$\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{f}+\mathbf{g})(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}),$$

The argument is the same in the vector case, but at one point in the proof, the triangle inequality is necessary. Likewise, the limit of the difference is the difference of the limits.

There are several different kinds of products that consider. We could have the product of a scalar function and a vector-valued function, $f(\mathbf{x})\,\mathbf{g}(\mathbf{x})$; the dot product of two vector-valued functions, $\mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x})$; or the cross product of two 3-dimensional vector-valued functions, $\mathbf{f}(\mathbf{x})\times\mathbf{g}(\mathbf{x})$. In each case, the product of the limits is the limit of the product.

Finally, the limit of the quotient is the quotient of the limits, provided the denominator does not approach 0 (and notice that the denominator has to be a scalar for the quotient to be defined).

All the proofs are like those in the scalar case, complicated in the cases of dot products and cross products by their definitions in terms of sums of products.

Thus, all the usual properties that hold for scalar limits also hold for vector limits.

Continuous functions. Recall from calculus of one variable that a function $f : \mathbf{R} \to \mathbf{R}$ is continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$

Also, we say a function is continuous if it is continuous at every number a in its domain.

Since limits preserve sums, differences, various kinds of products, and quotients, we know that the sum, difference, various products, and quotient of continuous functions are continuous (assuming, of course, for the quotient, that the denominator is a scalar, and it's not zero).

Most useful functions of one variable are continuous, but there are a few exceptions. For instance, step functions are continuous except at their steps, that is, where there are jump discontinuities. But polynomials, trig functions, power and root functions, logarithms, and exponential functions are all continuous. Furthermore, sums, differences, products, quotients, and compositions of continuous functions are continuous.

We use the analogous condition to define continuous multivariate functions. We say a function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at $\mathbf{a} \in \mathbf{R}^n$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

As you might expect, the useful multivariate functions are continuous. Also, sums, differences, products, quotients, and compositions of continuous multivariate functions are continuous.

Polynomials. You're familiar with polynomials in one variable x. These are functions $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

where the coefficients $a_d, a_{d-1}, ..., a_1, a_0$ are all scalar constants, and the leading coefficient a_d is not 0. The number d is the degree of the polynomial. Such a polynomial can also be written in summation notation as

$$f(x) = \sum_{k=0}^{d} a_k x^k.$$

Since polynomials are built from constants, x, addition, and multiplication, they're all continuous everywhere.

A polynomial in two variables x and y is a scalar-valued function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x,y) = \sum_{k=0}^{d} \sum_{l=0}^{d} a_{kl} x^k y^l.$$

For instance, $f(x,y) = 2x^2y^2 - 3x^2y + 5x^2 + 6xy^2 - xy - 7x + 4y^2 + 3y - 1$ is a polynomial of two variables. Again, since polynomials of two (or more) variables are built from constants, x, y, addition, and multiplication, they're all continuous everywhere. The *degree* of a polynomial of several variables is defined to be the maximum degree of all its monomial terms, and the *degree* of a monomial is defined to be the sum of the exponents of variables that appear in it. So, for instance, the polynomial $f(x,y)$ given above has degree 4 since the monomial $2x^2y^2$ has degree 4 and the rest of the monomials in f have lower degree.

Rational functions are quotients of polynomials, and whether they have one or more variables, they're continuous everywhere they're defined. (They're not defined when their denominators are 0.)

Component functions. Recall that a vector-valued function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is determined by its its m scalar-valued component functions $f_1, f_2, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$,

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})).$$

If each of these component functions has a limit,

$$\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i,$$

then $\mathbf{f}$ also has a limit

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = (L_1, L_2, \ldots, L_m),$$

and conversely.

From that observation, it follows that a vector-valued function $\mathbf{f}$ is continuous if and only if all of its component functions are continuous.

Some discontinuous functions. Recall from calculus that one of the typical reasons a limit $\lim_{x\to a} f(x)$ does not exist is that the left and right limits don't agree.

On a line, there are two ways to approach a point, from the left side and from the right side. But in higher dimensions, there are infinitely many ways to approach a point. Consider the example,

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

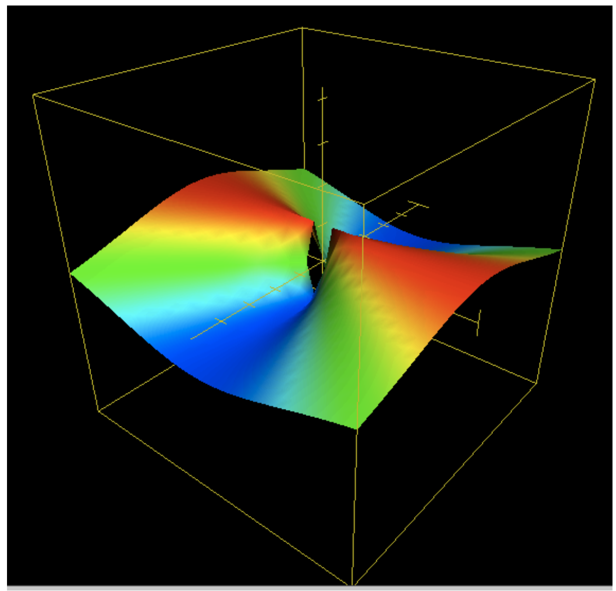

Figure 1: A singularity at $(x,y) = (0,0)$

The limit, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist because different values are approached depending on whether you approach the origin $(0,0)$ along the x-axis or the y-axis. Along the x-axis, where $y = 0$, $f(x,0) = 1$, but along the y-axis, where $x = 0$, $f(0,y) = -1$. And, of course $f(0,0)$ is not defined. There's a "singularity" when $(x,y) = (0,0)$.

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