

Polar coordinate conversion
 Math 131 Multivariate Calculus
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Change of coordinates. The most important use of the change of variables formula is for coordinate changes. And the most important change of coordinates is from rectangular to polar coordinates. We'll develop the formula for finding double integrals in polar coordinates. We'll show that the Jacobian to change to polar coordinates is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

The easiest way to remember the polar coordinate formulas is in terms of the area differential dA . For rectangular coordinates, $dA = dx dy$. But in polar coordinates, $dA = r dr d\theta$. That's because the Jacobian of the transformation is just r .

Polar coordinates. The equations to convert between rectangular and polar coordinates are

$$\begin{aligned} x &= r \cos \theta & r^2 &= x^2 + y^2 \\ y &= r \sin \theta & \tan \theta &= y/x \end{aligned}$$

The transformation between coordinate systems is

$$(x, y) = \mathbf{T}(r\theta) = (r \cos \theta, r \sin \theta).$$

Let's compute the Jacobian.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

Example 1. We'll transform the following integral given in rectangular coordinates to polar coordinates, and evaluate it.

$$\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2} dx dy$$

(This particular integral is important in statistics. It's related to the normal distribution.)

The integrand $e^{x^2+y^2}$ will become e^{r^2} . The area differential $dx dy$ will become $r dr d\theta$. So all that's left is to determine the limits of integration for polar coordinates.

Now, y varies from $-a$ to a , and x varies from 0 to $\sqrt{a^2 - y^2}$. The equation $x = \sqrt{a^2 - y^2}$ describes a semicircle, the right half of a circle of radius a . Therefore, the region under question is the right half of that circle. We can parameterize that region in terms of r and θ if we let r vary from 0 to a and θ vary from $-\pi/2$ to $\pi/2$. Thus, the integral in terms of polar coordinates is

$$\int_{-\pi/2}^{\pi/2} \int_0^a e^{r^2} r dr d\theta$$

First evaluate the inner integral. You can find it either by an explicit substitution where $u = r^2$ and $du = 2r dr$, or you might see the antiderivative right away.

$$\int_0^a e^{r^2} r dr = \frac{1}{2} e^{r^2} \Big|_0^a = \frac{1}{2} (e^{a^2} - 1)$$

We'll replace the inner integral by $e^{a^2} - 1$ and finish the integration.

$$\int_{-\pi/2}^{\pi/2} \frac{1}{2} (e^{a^2} - 1) d\theta = \frac{\pi}{2} (e^{a^2} - 1)$$

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