

Math 131 Multivariate Calculus
 Quiz Answers
 19 Mar 2010

Scale. 9–10 A, 7–8 B, 6 C. Median A.

Problem 1. [3] On arclength. The path $\mathbf{x} : \mathbf{R} \rightarrow \mathbf{R}^2$ given by

$$\mathbf{x}(t) = (x(t), y(t)) = (\cos^3 t, \sin^3 t)$$

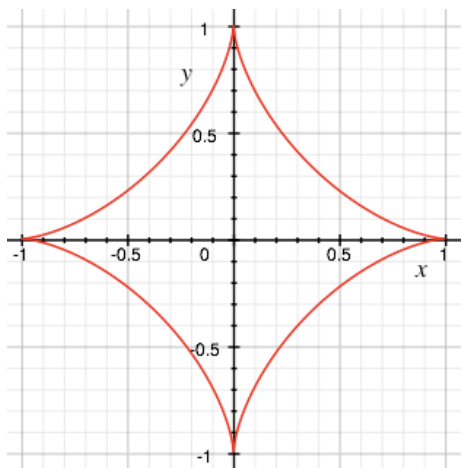
describes an astroid, that is, a star shaped figure. For t in the interval $[0, \frac{\pi}{2}]$, one fourth of the astroid is described. It is easy to see that

$$\mathbf{x}' = (-3 \sin t \cos^2 t, 3 \sin^2 t \cos t),$$

and

$$\|\mathbf{x}'\| = 3 \sin t \cos t = \frac{3}{2} \sin 2t.$$

Using that information, determine the arclength of this one-fourth astroid.



The arclength is

$$\begin{aligned} \int_0^{\pi/2} \|\mathbf{x}'\| dt &= \int_0^{\pi/2} \frac{3}{2} \sin 2t dt \\ &= -\frac{3}{4} \cos 2t \Big|_0^{\pi/2} \\ &= -\frac{3}{4} \cos \pi + \frac{3}{4} \cos 0 = \frac{3}{2} \end{aligned}$$

Incidentally, the astroid (a word meaning star) is a special case of a hypocycloid. A hypocycloid is the locus of a point on the circumference of a circle which is rolling around on the inside of a fixed circle. For the astroid, the ratio of the radius of the fixed circle to the rolling circle is 4:1. Astroids, also called tetracusps, were studied by Bernoulli in 1691. The astroid given in this problem has the equation $x^{2/3} + y^{2/3} = 1$ in rectangular coordinates, also $(x^2 + y^2 - 1)^3 + 27x^2y^2 = 0$. In polar coordinates, $r = ((\cos \theta)^{2/3} + (\sin \theta)^{2/3})^{-3/2}$.

Problem 2. [3] On flow lines. Recall that a flow line for a vector field \mathbf{F} is a path \mathbf{x} such that the velocity along the path is a vector in the vector field, that is, $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$. Verify that the path $\mathbf{x}(t) = (\sin t, \cos t, 2t)$ is a flow line for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 2\mathbf{k} = (y, -x, 2)$.

Since $(x, y, z) = (\sin t, \cos t, 2t)$, therefore $\mathbf{x}'(t) = (x', y', z') = (\cos t, -\sin t, 2)$ while $\mathbf{F}(\mathbf{x}(t)) = (y, -x, 2) = (\cos t, -\sin t, 2)$.

Problem 3. [4; 2 points each part] On divergence and curl of vector fields.

a. Give an example of a vector field with a nonzero divergence, and compute its divergence.

The divergence of a vector field $\mathbf{F} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is defined as

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

Almost any vector field you name will have a nonzero divergence. For instance, $\mathbf{F}(x, y, z) = (x, 0, 0)$ has divergence

$$\operatorname{div} \mathbf{F}(x, y, z) = 1 + 0 + 0 = 1.$$

b. Give an example of a vector field with a nonzero curl, and compute its curl.

The curl of a vector field $\mathbf{F} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is defined as

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \times (F_1, F_2, F_3) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

Almost any vector field you name will have a nonzero curl, but not all of them. The example given for part a has a 0 curl. But $\mathbf{F}(x, y, z) = (y, 0, 0)$ has curl

$$\operatorname{curl} \mathbf{F}(x, y, z) = (0 - 0, 0 - 0, 0 - 1) = (0, 0, -1).$$