Math 131 Multivariate Calculus

Quiz Answers

26 Apr 2010

Scale. 9–10 A, 7–8 B. Median 9.

Problem 1. [6] On path integrals. Consider the path parameterized by $\mathbf{x}(t) = (\cos t, \sin t, \ln t)$ for $1 \leq t \leq 5$. Write down integrals for the following. The only variable that should appear in the integrals is t. Do not evaluate the integrals.

In the various parts of this problem, you'll need the velocity and speed.

$$\mathbf{x}'(t) = (-\sin t, \cos t, 1/t)$$

$$\|\mathbf{x}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1/t^2} = \sqrt{1 + 1/t^2}$$

a. Write down an integral giving the length of the path.

The length of the path is the same thing as the scalar line integral of the constant 1.

$$\begin{aligned} \int_{\mathbf{x}} 1\, ds &= \int_a^b \|\mathbf{x}'(t)\|\, dt \\ &= \int_1^5 \sqrt{1 + 1/t^2}\, dt \end{aligned}$$

b. Write down a scalar line integral for the scalar field $f(x, y, z) = x^2 + y^2 + e^z$.

$$\begin{aligned} \int_{\mathbf{x}} f\, ds &= \int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\, dt \\ &= \int_1^5 f(\cos t, \sin t, \ln t)\sqrt{1 + 1/t^2}\, dt \\ &= \int_1^5 (\cos^2 t + \sin^2 t + t)\sqrt{1 + 1/t^2}\, dt \\ &= \int_1^5 (1 + t)\sqrt{1 + 1/t^2}\, dt \end{aligned}$$

c. Write down a vector line integral for the vector field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + xy\mathbf{j} + 2\mathbf{k}$.

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\ &= \int_1^5 \mathbf{F}(\cos t, \sin t, \ln t) \cdot (-\sin t, \cos t, 1/t)\, dt \\ &= \int_1^5 ((\ln t)^2, \cos t\, \sin t, 2) \cdot (-\sin t, \cos t, 1/t)\, dt \\ &= \int_1^5 (-(\ln t)^2 \sin t + \cos^2 t\, \sin t + 2/t)\, dt \end{aligned}$$

Problem 2. [4] On Green's theorem.

Let D be the disk $x^2 + y^2 \leq 4$. Use Green's theorem to convert the line integral

$$\oint_{\partial D} -x^2 y\, dx + xy^2\, dy$$

into a double integral. Leave the D in the double integral; don't put in limits of integration for x and y, and don't evaluate the resulting double integral.

Green's theorem says

$$\oint_{\partial D} M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

Here, $M = -x^2 y$ and $N = xy^2$. Therefore, the double integral is

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial\, y}\right) dx\, dy = \iint_D (y^2 + x^2)\, dx\, dy.$$

If you went on to evaluate this integral, you probably wouldn't even find the limits of integration for x and y. Instead you'd use polar coordinates to convert it to

$$\int_0^{2\pi} \int_0^2 r^2\, r\, dr\, d\theta$$

where the extra r in the integral is the Jacobian for converting to polar coordinates.