CLARK UNIVERSITY 1887

Section 2.1 selected answers

Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises 1–7, 10, 15–21 odd, 31, 39.

2. Let $g : \mathbf{R}^2 \to \mathbf{R}$ be given by $g(x, y) = 2x^2 + 3y^2 - 7$.

(a) Find the domain and range of g.

Since g is always defined, its domain is all of $\mathbf{R}^2$. Since x^2 and y^2 are always each ≥ 0, the minimum value of g is -7. Therefore, its range is the interval $[-7, \infty)$.

(b) Restrict the domain of g to make it one-to-one.

The problem is that g has the same values for many values of (x, y). For instance, on the entire ellipse $2x^2 + 3y^2 = 7$, g has the value 0. One solution, out of many possible solutions, is to make $y = 0$ and $x \geq 0$. When that restriction is made, g becomes one-to-one.

(c) Restrict the codomain of g to make it onto.

You can always restrict the codomain to the range to make it onto.

4. Find the domain and range of $f(x, y) = \ln(x + y)$.

In order for ln to be defined, $x + y$ has to be positive. In other words the domain of f is the set of (x, y) such that $x + y > 0$.

Within that domain $x + y$ can be any positive number, so the range of f is just the range of ln, and the range of ln is all of $\mathbf{R}$.

6. Find the domain and range of

$$g(x, y, z) = \frac{1}{\sqrt{4 - x^2 - y^2 - z^2}}.$$

In order for g to be defined, there are two requirements. First, $4 - x^2 - y^2 - z^2$ has to be ≥ 0 so that its square root is defined. Second, $\sqrt{4 - x^2 - y^2 - z^2}$ cannot be 0, since it's in a denominator. Together, these conditions require $x^2 + y^2 + z^2 < 4$. If you want to write that in set notation, it looks like

$$\{(x, y, z) \mid x^2 + y^2 + z^2 < 4\}.$$

This set can be described geometrically as the open ball in $\mathbf{R}^3$ of radius 2 about the origin.

Next, to determine the range, note that the denominator is a positive number, but it can't be larger than 2 since the maximum of $\sqrt{4 - x^2 - y^2 - z^2}$ is $\sqrt{2}$. Since the denominator is between 0 and 2, its reciprocal is between $\frac{1}{2}$ and ∞. Thus, the range is the half-open interval $[\frac{1}{2}, \infty)$.

10. Let $\mathbf{f} : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $\mathbf{f}(\mathbf{x}) = \mathbf{x} + 3\mathbf{j}$. Write out the component functions of $\mathbf{f}$ in terms of the components of the vector $\mathbf{x}$.

Let $\mathbf{x} = (x, y, z)$ as usual. Then

$$\mathbf{f}(\mathbf{x}) = \mathbf{x} + 3\mathbf{j} = (x, y, z) + 3(0, 1, 0) = (x, y + 3, z).$$

Therefore, the component functions are $f_1(x, y, z) = x$, $f_2(x, y, z) = y + 3$, and $f_3(x, y, z) = z$.

19. Determine level curves and sketch the graph of $f(x, y) = xy$.

The curve of height c is the solution set for $xy = c$. If $c \neq 0$, then the curve is a rectangular hyperbola whose asymptotes are the x- and y-axes. When $c = 0$, the level curve is the union of the x- and y-axes. The surface is called a hyperbolic paraboloid.

31. Given a function $f(x, y)$, can two different level curves of f intersect? Why or why not?

If the level curve for height c, which is $f(x, y) = c$, intersects the level curve for height d, which is $f(x, y) = d$, then $c = d$. Therefore, the level curves for two distinct heights cannot intersect.

39. Is it possible to find a function $f(x,y)$ so that the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ is the graph $z = f(x,y)$?

To be the graph of a function $f(x,y)$, for a given vector (x,y) there can be at most one value z such that (x,y,z) lies on the surface. But for this surface, there will be two, both z and $-z$. So this surface is not the graph of a function. It is, however, the union of the graphs of two functions

$$z = \pm\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

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