Section 2.6 selected answers

Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises 2, 3, 12, 13, 15, 16.

2. Calculate the directional derivative of $f(x,y) = e^y \sin x$ at the point $\mathbf{a} = (\pi/3, 0)$ in the direction $\mathbf{u} = (3\mathbf{i} - \mathbf{j})/\sqrt{10}$.

First find the gradient.

$$\begin{aligned}\nabla f(x,y) &= \left(\frac{\partial}{\partial x} e^y \sin x, \frac{\partial}{\partial y} e^y \sin x\right) \\ &= (e^y \cos x, e^y \sin x)\end{aligned}$$

Then the directional derivative is the dot product

$$\begin{aligned}D_{\mathbf{u}} f(\mathbf{a}) &= \nabla f(\mathbf{a}) \cdot \mathbf{u} \\ &= (e^y \cos x, e^y \sin x) \cdot (3, -1)/\sqrt{10} \\ &= (\cos(\pi/3), \sin(\pi/3)) \cdot (3, -1)/\sqrt{10} \\ &= (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}) \cdot (3, -1))/\sqrt{10} \\ &= \frac{3 - \sqrt{3}}{2\sqrt{10}}\end{aligned}$$

12. A ladybug is crawling on graph paper. She is at the point $(3, 7)$ and notices that if she moves in the $\mathbf{i}$-direction, the temperature increases at a rate of 3 deg/cm. If she moves in the $\mathbf{j}$-direction, she finds that her temperature decreases at a rate of 2 deg/cm. In what direction should she move if

a. she wants to warm up most rapidly?

That is the direction of the gradient. The gradient is $\nabla f = (3, -2)$ since the partial derivatives are 3 and -2. To make that a direction, normalize it to a unit vector

$$\mathbf{u} = \frac{(3,-2)}{\|(3,-2)\|} = \left(\frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right)$$

b. she wants to cool off most rapidly?

That is the opposite direction, the negation of $\mathbf{u}$.

c. desires her temperature not to change?

That is a direction orthogonal to $\mathbf{u}$, and that would be $\pm\left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right)$.

15. Igor, the inchworm, is crawling along graph paper in a magnetic field. The intensity of the field at the point (x, y) is given by $M(x,y) = 3x^2 + y^2 + 5000$. If Igor is at the point $(8, 6)$, describe the curve along which he should travel if he wishes to reduce the field intensity as rapidly as possible.

The gradient of M at a point (x, y) is $\nabla M(x,y) = (6x, 2y)$. To reduce M as much as possible, the opposite direction $(-6x, -2y)$ should be taken.

So, we're looking for a path $\mathbf{x}(t) = (x(t), y(t))$ whose derivative is $(-6x(t), -2y(t))$. In other words, we need two functions $x(t)$ and $y(t)$ such that

$$\begin{aligned}\frac{dx}{dt} &= = -6x(t) \\ \frac{dy}{dt} &= -2y(t)\end{aligned}$$

Each is a differential equation, and we can solve them separately to get $x(t) = Ae^{-6t}$ and $y(t) = Be^{-2t}$, where A and B are constants. At $t = 0$, the position is $(x(0), y(0)) = (8, 6)$, so $8 = Ae^0$ and $6 = Be^0$, so we conclude that $A = 8$ and $B = 6$. Thus, $\mathbf{x}(t) = (8e^{-6t}, 6e^{-2t})$. To see what curve this path takes, let's eliminate t from the pair of equations $x = 8e^{-6t}$ and $y = 6e^{-2t}$. Note $e^{-2t} = y/6$, so $x = 8(y/6)^3$. Thus, $x = y^3/27$.

Alternatively, to solve the pair of differential equations, you could eliminate the variable t since

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{y}{3x}.$$

Separate of variables to get $\dfrac{dy}{y} = \dfrac{dx}{3x}$ and, then integrating, $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{3x}$, so $\ln|y| = \frac{1}{3}\ln|x| + C$, which gives us, writing A for e^C, $|y| = A|x|^{1/3}$.

That describes the curves of steepest descent as a family of parabolas parameterized by the real number A:

$$x = Ay^3.$$

Then determine A using the fact that the curve passes through the point $(8, 6)$.

16. Find an equation for the tangent plane to the surface $x^3 + y^3 + z^3 = 7$ at the point $(x_0, y_0, z_0) = (0, -1, 2)$.

The tangent plane has the equation

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0.$$

Since $f(\mathbf{x}) = x^3 + y^3 + z^3$, therefore

$$\nabla f(\mathbf{x}) = (3x^2, 3y^2, 3z^2),$$

and when that is evaluated at the point, we get

$$\nabla f(\mathbf{x}_0) = (0, 3, 12).$$

Thus, this tangent plane has the equation

$$(0, 3, 12) \cdot (x - 0, y - 3, z - 12) = 0,$$

which can be rewritten as $3y + 12z = 153$.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`