Section 3.1 answers

Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises from section 3.1: 1–4, 7–8, 15–16.

1–4. Sketch the image of the following paths using arrows to indicate the direction in which the parameter increases.

2. $\mathbf{x}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$.

So, $x = e^t$ while $y = e^{-t}$. Eliminate t to determine the underlying curve. You get $y = 1/x$, a hyperbola. Next, which part of the hyperbola does this path lie on, and which direction is the path going? Since both x and y are positive, the path lies on the first-quadrant branch of this hyperbola. Also, as the time t increases, so does x, but y is decreasing. Therefore, the path goes to the right and downward on this branch of the hyperbola.

4. $x = 3\cos t$ and $y = 2\sin 2t$ where $0 \le t \le 2\pi$.

As t increases from 0 to 2π, the value $\cos t$ goes from 1 to 0 to -1 to 0 and back to 1. Therefore, $x = 3\cos t$ goes from 3 to 0 to -3 to 0 and back to 3. At the same time, $\sin 2t$ goes through two sine cycles: 0 to 1 to 0 to -1 to 0 to 1 to 0 to -1. Therefore $y = 2\sin 2t$ runs through the two cycles: 0 to 2 to 0 to -2 to 0 to 2 to 0 to -2. If you sketch this figure, it looks something like a big ∞ figure, bounded by a rectangle where $-3 \le x \le 3$ and $-2 \le y \le 2$.

If you want, you can find the equation of the curve in terms of x and y as follows. First note that $y = 2\sin 2t = 4\sin t\cos t$, by the double angle formula for sine. Therefore,

$$\begin{aligned} y^2 &= 16\sin^2 t\cos^2 t \\ &= 16\left(1-\left(\frac{x}{3}\right)^2\right)\left(\frac{x}{3}\right)^2 \\ &= \frac{16}{81}(9-x^2)x^2 \end{aligned}$$

8. Calculate the velocity, speed and acceleration of the path

$$\mathbf{x}(t) = 5\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}.$$

Since the position at time t is

$$\mathbf{x}(t) = (x,y) = (5\cos t, 3\sin t),$$

therefore the velocity is

$$\mathbf{v}(t) = \mathbf{x}'(t) = (x',y') = (-5\sin t, 3\cos t),$$

and the acceleration is

$$\mathbf{a}(t) = \mathbf{x}''(t) = (x'',y'') = (-5\cos t, -3\sin t).$$

The speed is

$$\|\mathbf{x}'(t)\| = \sqrt{(x')^2+(y')^2} = \sqrt{25\cos^2 t + 9\sin^2 t}.$$

16. Find the equation for the line tangent to the path

$$\mathbf{x}(t) = 4\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j} + 5t\,\mathbf{k}$$

at the time $t = \pi/3$.

You can see from the formula for this path that it's a variant of the helix we studied in class. The coefficients 4 and 3 mean that it is an elliptical helix instead of a circular helix, and the minus sign in front of the 3 means it's traveling clockwise around the z-axis instead of counterclockwise.

In general, the equation of the tangent line can be given parametrically as

$$\mathbf{x}_0 + s\mathbf{v}_0$$

where $\mathbf{x}_0$ is the point on the line at the particular time t_0, $\mathbf{v}_0$ is the velocity at that time, and s is a scalar parameter.

In our case

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{x}(\tfrac{\pi}{3}) \\ &= (4\cos\tfrac{\pi}{3}, -3\sin\tfrac{\pi}{3}, 5(\tfrac{\pi}{3})) \\ &= (2, -\tfrac{3\sqrt{3}}{2}, \tfrac{5\pi}{3}). \end{aligned}$$

Since

$$\mathbf{v}(t) = \mathbf{x}'(t) = (-4\sin t, -3\cos t, 5),$$

therefore

$$\begin{aligned} \mathbf{v}_0 &= \mathbf{v}(\tfrac{\pi}{3}) \\ &= (-4\sin\tfrac{\pi}{3}, -3\cos\tfrac{\pi}{3}, 5) \\ &= (2\sqrt{3}, -\tfrac{3}{2}, 5). \end{aligned}$$

Thus, a parametric equation for the tangent line is

$$\mathbf{x}_0 + s\mathbf{v}_0 = (2, -\tfrac{3\sqrt{3}}{2}, \tfrac{5\pi}{3}) + s(2\sqrt{3}, -\tfrac{3}{2}, 5).$$

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