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Section 3.2 selected answers

Math 131 Multivariate Calculus

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Exercises from section 3.2: 1, 2, 4, 8, 11, 18. Selected answers

2. Calculate the length of the path

$$\mathbf{x}(t) = \left(t^2, \tfrac{2}{3}(2t+1)^{3/2}\right)$$

where $0 \leq t \leq 4$.

The velocity is

$$\mathbf{x}'(t) = \left(2t, 2(2t+1)^{1/2}\right).$$

The length of the path is the integral of the speed:

$$\begin{aligned}\int_0^4 \|\mathbf{x}'(t)\|\, dt &= \int_0^4 \|(2t, 2(2t+1)^{1/2}\|\, dt \\ &= \int_0^4 \sqrt{4t^2+4(2t+1)}\, dt \\ &= \int_0^4 2\sqrt{t^2+2t+1}\, dt \\ &= \int_0^4 2(t+1)\, dt \\ &= (t^2+2t)\Big|_0^4 = 24\end{aligned}$$

4. Calculate the length of the path

$$\mathbf{x}(t) = (7, t, t^2)$$

where $1 \leq t \leq 3$.

This path is part of a parabola in the plane $x = 7$ in $\mathbf{R}^3$. There aren't many curves whose lengths we can compute exactly, but this is one.

The length of the path is

$$\begin{aligned}\int_1^3 \|\mathbf{x}'(t)\|\, dt &= \int_1^3 \|(0,1,2t)\|\, dt \\ &= \int_1^3 \sqrt{1+4t^2}\, dt\end{aligned}$$

You can use integral tables or techniques of integration to finish this exercise. One way to find this integral is by the substitution $2t = \tan\theta$. Then $\sqrt{1+4t^2} = \sqrt{1+\tan^2\theta} = \sec\theta$, and $2dt = \sec^2\theta\, d\theta$. The integral becomes

$$\int \sec^3\theta\, d\theta$$

which with integration by parts gives

$$\tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\ln(\sec\theta+\tan\theta),$$

and putting this back in terms of t gives

$$t\sqrt{1+4t^2} + \tfrac{1}{2}\ln\left(\sqrt{1+4t^2}+2t\right).$$

So the definite integral equals

$$t\sqrt{1+4t^2} + \tfrac{1}{2}\ln\left(\sqrt{1+4t^2}+2t\right)\Big|_1^3 =$$

$$3\sqrt{37} + \tfrac{1}{2}\ln\left(\sqrt{37}+6\right) - \sqrt{5} - \tfrac{1}{2}\ln\left(\sqrt{5}+2\right)$$

8. $\mathbf{x}(t) = (2t\cos t, 2t\sin t, 2\sqrt{2}t^2)$.

$$\mathbf{x}' = (2\cos t - 2t\sin t, 2\sin t + 2t\cos t, 4\sqrt{2})$$

$$\|\mathbf{x}'\| = \sqrt{4+4t^2+32t^2} = 2\sqrt{1+9t^2}$$

The length of the path is

$$\int_0^3 2\sqrt{1+9y^2}\, dt$$

which can be found by a trig sub or tables.

Math 131 Home Page at http://math.clarku.edu/~djoyce/ma131/