CLARK UNIVERSITY 1887

Section 3.3 selected answers

Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises from section 3.3: 1, 4, 9, 10, 19–21, 24, 26.

4. Sketch the vector field $\mathbf{F}(x, y) = (x, x^2)$.

The first thing you notice about this vector field is that the values of $\mathbf{F}$ do not depend on y, but only on x. That means that all the vectors along a vertical line are parallel. So, all you have to do is draw the vectors along one horizontal line, say the x-axis, and then translate them vertically to get vectors to fill the plane.

Now, for x positive, both coordinates of (x, x^2) are positive, so those arrows point up and to the right, and the larger x is, the longer the arrows should be and the more vertical they should be. But if x is positive near 0, the the arrows should be short and nearly horizontal.

A similar argument applies for negative x. Vectors should point up and to the left. If x is near $-\infty$ they are long and almost vertical, but if x is near 0, then they are short and almost horizontal.

10. Sketch the vector field on $\mathbf{R}^3$ for $\mathbf{F} = (y, -x, 2)$.

Forget the third coordinate for a minute. Pretend $\mathbf{F}(x, y) = (y, -x)$. This we know as a 'circular' vector field swirling around the origin in the clockwise direction. The vector you draw at (x, y) has the same length as the vector (x, y) but is at right angles to it.

Now add the third coordinate and see what happens. $\mathbf{F}(x, y, z) = (y, -x, 2)$. The vector $(y, -x, 2)$ you draw at (x, y, z) is not just the translate of the vector we just drew at (x, y), but it has a vertical component as well; its end should be lifted up 2 units. So, this is some sort of 'helical' vector field.

20. Calculate the flow line $\mathbf{x}(t)$ of the vector field $\mathbf{F}(x, y) = (-x, y)$ if $\mathbf{x}(0) = (2, 1)$.

The flow line, also called a flow path, $\mathbf{x}(t)$ has to have its derivatives in the vector field, that is

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)).$$

In this exercise, that says $(x'(t), y'(t) = (-x(t), y(t))$. That gives us the two differential equations

$$\begin{aligned} x' &= -x \\ y' &= y \end{aligned}$$

We can solve these differential equations to get $x(t) = Ae^{-t}$ and $y(t) = Be^t$, where A and B are constants yet to be determined. Now, we're told that $(x, y)(0) = (2, 1)$, that is $x(0) = 2$ and $y(0) = 1$, therefore, $A = 2$ and $B = 1$. Thus, the flow line is

$$\mathbf{x}(t) = (x(t), y(t)) = (2e^{-t}, e^t).$$

(If you eliminate t from this equation, you get $xy = 2$, so the flow is along a hyperbola.)

24. Consider the vector field

$$\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}.$$

a. Show that $\mathbf{F}$ is a gradient field.

We need to find a scalar function $f(x, y, z)$ such that $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = -3$. One such function is

$$f(x, y, z) = x^2 + y^2 - 3z.$$

(Any other would differ from this one by a constant.)

b. Describe the equipotential surfaces of $\mathbf{F}$ in words and with sketches.

An equipotential surface of $\mathbf{F}$ is a level surface of f, that is, a surface whose equation is $x^2 + y^2 - 3z = c$ for some constant c. This is some quadric surface since the equation is a quadratic equation. In fact, it's a paraboloid whose axis is the z-axis. The vertex of this paraboloid is at the point $-c/3$.

That's one equipotential surface. They're all translates of each other in the z-direction.

26. A flow of $\mathbf{F}$ is a differentiable function $\phi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{F}(\phi(\mathbf{x}, t))$$

and

$$\phi(\mathbf{x}, 0) = \mathbf{x}.$$

In other words, it includes all the flow lines of $\mathbf{F}$ in one function parameterized by $\mathbf{x}$.

Verify that $\phi : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ given by

$$\phi(x, y, t) = \left(\frac{x+y}{2}e^t + \frac{x-y}{2}e^{-t}, \frac{x+y}{2}e^t + \frac{y-x}{2}e^{-t}\right)$$

is a flow of the vector field $\mathbf{F}(x, y) = (y, x)$.

First, the partial derivative of the flow ϕ with respect to t is

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \left(\frac{x+y}{2}e^t - \frac{x-y}{2}e^{-t}, \frac{x+y}{2}e^t - \frac{y-x}{2}e^{-t}\right).$$

Next,

$$\mathbf{F}(\phi(\mathbf{x}, t)) = \left(\frac{x+y}{2}e^t + \frac{y-x}{2}e^{-t}, \frac{x+y}{2}e^t + \frac{x-y}{2}e^{-t}\right).$$

But those are equal, so $\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{F}(\phi(\mathbf{x}, t))$. Also, $\phi(\mathbf{x}, 0) = \mathbf{x}$, since

$$\phi(x, y, 0) = \left(\frac{x+y}{2} + \frac{x-y}{2}, \frac{x+y}{2} + \frac{y-x}{2}\right) = (x, y).$$

Thus, ϕ is a flow for $\mathbf{F}$.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`