CLARK UNIVERSITY 1887

Section 3.4 selected answers

Math 131 Multivariate Calculus

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Exercises from section 3.4: 1–4, 7–10, 13, 28a.

1. Calculate the divergence of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$.

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} y^2 \\ &= 2x + 2y \end{aligned}$$

2. Calculate the divergence of $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$.

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial}{\partial x} y^2 + \frac{\partial}{\partial y} x^2 \\ &= 0 \end{aligned}$$

Note that since the divergence is 0, this vector field is incompressible.

4. Calculate the divergence of

$$\mathbf{F} = z\cos(e^{y^2})\,\mathbf{i} + x\sqrt{z^2+1}\,\mathbf{j} + e^{2y}\sin 3x\,\mathbf{k}.$$

$$\begin{aligned} &\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} \\ = &\ \frac{\partial}{\partial x} z\cos(e^{y^2}) + \frac{\partial}{\partial y} x\sqrt{z^2+1} + \frac{\partial}{\partial z} e^{2y}\sin 3x \\ = &\ 0 + 0 + 0 = 0 \end{aligned}$$

Another incompressible vector field.

8. Find the curl of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\begin{aligned} \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) \\ &= (0, 0, 0) \end{aligned}$$

Note that since this curl is $\mathbf{0}$, the radial vector field $\mathbf{F}(x, y, z) = (x, y, z)$ is irrotational.

10. Find the curl of

$$\mathbf{F} = (\cos yz - x)\,\mathbf{i} + (\cos xz - y)\,\mathbf{j} + (\cos xy - z)\,\mathbf{k}.$$

I interpret $\cos yz - x$ as $(\cos(yz)) - x$. Generally, named functions like cos and log have lower precedence than multiplication and division, but higher precedence than addition and subtraction.

$$\begin{aligned} \text{curl } \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= (-x\sin xy + x\sin xz, \\ &\quad -y\sin yz + y\sin xy \\ &\quad -z\sin xz + z\sin yz) \end{aligned}$$

13. Can you tell in what portions of $\mathbf{R}^2$ the vectors field shown in figures 3.42–3.45 have positive divergence or negative divergence?

In figure 3.42 the vectors are all pointing away from the origin and they're getting longer as you follow them along. So the divergence is positive everywhere.

In figure 3.43 the vectors are all pointing toward the origin and as you follow them along, the vectors get shorter. So the divergence is negative everywhere.

In figure 3.44, for $x > 0$ the vectors point away from the origin and they're getting longer, so the divergence is positive there. But for $x < 0$ they point toward the origin and they're getting shorter, so the divergence is negative there.

It's hard to tell what's going on in figure 3.45. Above the x-axis the vectors are getting shorter while below it, the vectors are getting longer. But length is not the whole story since the vectors are not parallel. For instance, in part of the plane the vectors are geting longer but they're also getting closer together. That happens below the x-axis between the lines $y = x$ and $y = -x$. When a vector field is made of parallel vectors getting longer, the divergence is positive; when a vector field is made of converging vectors of the same length, the divergence is negative; here we have converging vectors getting longer.

28a. The Laplacian operator, denoted ∇^2, is the second-order partial differential operator defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Explain why it makes sense to think of ∇^2 as $\nabla \cdot \nabla$.

Well, $\nabla \cdot \nabla$ is $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ which is $\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}$, and that's $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

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`http://math.clarku.edu/~djoyce/ma131/`