CLARK UNIVERSITY 1887

Section 4.1 selected answers

Math 131 Multivariate Calculus

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Exercises from section 4.1: 1, 2, 8, 9, 11, 19, 20, 24.

1. Find the Taylor polynomial p_4 for $f(x) = e^{2x}$ at $a = 0$.

It's probably easiest to start with a table of the first 4 derivatives of f and their values at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	e^{2x}	1
1	$2e^{2x}$	2
2	$4e^{2x}$	4
3	$8e^{2x}$	8
4	$16e^{2x}$	16

Therefore,

$$\begin{aligned} p_4(x) &= f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 \\ &= 1 + 2x + \tfrac{4}{2}x^2 + \tfrac{8}{3!}x^3 + \tfrac{16}{4!}x^4 \\ &= 1 + 2x + 2x^2 + \tfrac{4}{3}x^3 + \tfrac{2}{3}x^4 \end{aligned}$$

2. Find the Taylor polynomial p_3 for $f(x) = \ln(1+x)$ at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-1(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2

Therefore,

$$\begin{aligned} p_3(x) &= f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 \\ &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 \end{aligned}$$

8. Find the first-and second-order Taylor polynomials for the function $f(x,y) = 1/(x^2+y^2+1)$ at $\mathbf{a} = (a,b) = (0,0)$.

Start by making a table of the first and second derivatives. (Note that for all these functions, the mixed partial derivatives don't depend on the order of differentiation.)

$f = (x^2+y^2+1)^{-1}$	$f(\mathbf{a}) = 1$
$f_x = -2x(x^2+y^2+1)^{-2}$	$f_x(\mathbf{a}) = 0$
$f_y = -2y(x^2+y^2+1)^{-2}$	$f_y(\mathbf{a}) = 0$
$f_{xx} = (6x^2-2y^2-2)(x^2+y^2+1)^{-3}$	$f_{xx}(\mathbf{a}) = -2$
$f_{xy} = -8xy(x^2+y^2+1)^{-3}$	$f_{xy}(\mathbf{a}) = 0$
$f_{yy} = (6y^2-2x^2-2)(x^2+y^2+1)^{-3}$	$f_{yy}(\mathbf{a}) = -2$

Note that since the first partial derivatives are 0 at the point $\mathbf{a}$, there fore $\mathbf{a}$ is a critical point for the function. The Taylor polynomials are

$$\begin{aligned} p_1(x,y) &= f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b) \\ &= 1 \\ p_2(x,y) &= p_1(x,y) \\ &\quad + \tfrac{1}{2}f_{xx}(a,b)\,(x-a)^2 \\ &\quad + f_{xy}(a,b)\,(x-a)(y-b) \\ &\quad + \tfrac{1}{2}f_{yy}(a,b)\,(y-b)^2 \\ &= 1 - x^2 - y^2. \end{aligned}$$

9. Find the first-and second-order Taylor polynomials for the same function $f(x,y) = 1/(x^2+y^2+1)$, but at $\mathbf{a} = (a,b) = (1,-1)$.

Most of the work is already done in problem 8. The only difference is that all the partial derivatives have to be evaluated at $(1,-1)$ instead of $(0,0)$.

$$\begin{aligned}
p_1(x,y) &= f(a,b)+f_x(a,b)\,(x-a)+f_y(a,b)\,(y-b)\\
&= \tfrac{1}{3}-\tfrac{2}{9}(x-1)+\tfrac{2}{9}(y+1)\\
p_2(x,y) &= p_1(x,y)\\
&\quad+\tfrac{1}{2}f_{xx}(a,b)\,(x-a)^2\\
&\quad+f_{xy}(a,b)\,(x-a)(y-b)\\
&\quad+\tfrac{1}{2}f_{yy}(a,b)\,(y-b)^2\\
&= \tfrac{1}{3}-\tfrac{2}{9}(x-1)+\tfrac{2}{9}(y+1)\\
&\quad+\tfrac{1}{2}\tfrac{2}{27}(x-1)^2\\
&\quad+\tfrac{8}{27}(x-1)(y+1)\\
&\quad+\tfrac{1}{2}\tfrac{2}{27}(y+1)^2
\end{aligned}$$

19. Calculate the Hessian matrix $Hf(\mathbf{a})$ for the function $f(x,y,z)=x^3+x^2y-yz^2+2z^3$ at the point $\mathbf{a}=(a,b,c)=(1,0,1)$.

We'll need all the first- and second-order partial derivatives. Here's the first.

$f\ =x^3+x^2y-yz^2+2z^3$	$f(\mathbf{a})=3$
$f_x\ =3x^2+2xy$	$f_x(\mathbf{a})=3$
$f_y\ =x^2-z^2$	$f_y(\mathbf{a})=0$
$f_z\ =-2yz+6z^2$	$f_z(\mathbf{a})=6$

We can put the second-order partial derivatives in the square array Hf.

$$\begin{aligned}
Hf &= \begin{bmatrix} f_{xx} & f_{xy} & f_{xz}\\ f_{yx} & f_{yy} & f_{yz}\\ f_{zx} & f_{zy} & f_{zz}\end{bmatrix}\\
&= \begin{bmatrix} 6x+2y & 2x & 0\\ 2x & 0 & -2z\\ 0 & -2z & -2y+12z\end{bmatrix}
\end{aligned}$$

When we evaluate these partial derivatives at $\mathbf{a}=(1,0,1)$ we get the desired Hessian:

$$\begin{aligned}
Hf(\mathbf{a}) &= \begin{bmatrix} f_{xx}(\mathbf{a}) & f_{xy}(\mathbf{a}) & f_{xz}(\mathbf{a})\\ f_{yx}(\mathbf{a}) & f_{yy}(\mathbf{a}) & f_{yz}(\mathbf{a})\\ f_{zx}(\mathbf{a}) & f_{zy}(\mathbf{a}) & f_{zz}(\mathbf{a})\end{bmatrix}\\
&= \begin{bmatrix} 6 & 2 & 0\\ 2 & 0 & -2\\ 0 & -2 & 12\end{bmatrix}
\end{aligned}$$

20. Calculate the Hessian matrix $Hf(\mathbf{a})$ for the function $f(x,y,z)=e^{2x-3y}\sin 5z$ at the point $\mathbf{a}=(a,b,c)=(0,0,0)$.

$$\begin{aligned}
f_x &= 2e^{2x-3y}\sin 5z\\
f_y &= -3e^{2x-3y}\sin 5z\\
f_z &= 5e^{2x-3y}\cos 5z
\end{aligned}$$

$$Hf=\begin{bmatrix} f_{xx} & f_{xy} & f_{xz}\\ f_{yx} & f_{yy} & f_{yz}\\ f_{zx} & f_{zy} & f_{zz}\end{bmatrix}=$$

$$\begin{bmatrix} 4e^{2x-3y}\sin 5z & -6e^{2x-3y}\sin 5z & 10e^{2x-3y}\cos 5z\\ -6e^{2x-3y}\sin 5z & 9e^{2x-3y}\sin 5z & -15e^{2x-3y}\cos 5z\\ 10e^{2x-3y}\cos 5z & -15e^{2x-3y}\cos 5z & -25e^{2x-3y}\sin 5z\end{bmatrix}$$

When we evaluate these partial derivatives at $\mathbf{a}=(0,0,0)$ we get the desired Hessian:

$$Hf(\mathbf{a})=\begin{bmatrix} 0 & 0 & 10\\ 0 & 0 & -15\\ 10 & -15 & 0\end{bmatrix}$$

24. For $f(x,y,z)=x^3+x^2y-yz^2+2z^3$ at the point $\mathbf{a}=(a,b,c)=(1,0,1)$ as in exercise 19, express the second-order Taylor polynomial $p_2(x,y,z)$ using the derivative matrix and the Hessian matrix using the formula

$$p_2(\mathbf{x})=f(\mathbf{a})+Df(\mathbf{a})\mathbf{h}+\tfrac{1}{2}\mathbf{h}^T Hf(\mathbf{a})\mathbf{h}.$$

In this formula, $\mathbf{a}=(1,0,1)$, $f(\mathbf{a})=3$, and

$$\mathbf{h}=\mathbf{x}-\mathbf{a}=(h_1,h_2,h_3)=(x-1,y,z-1).$$

From exercise 19, we have

$$Df(\mathbf{a})=\nabla f(\mathbf{a})=(f_x(1),f_y(0),f_z(1))=(3,0,6).$$

Therefore,

$$\begin{aligned}
Df(\mathbf{a})\mathbf{h} &= f_x(\mathbf{a})h_1+f_y(\mathbf{a})h_2+f_z(\mathbf{a})h_3\\
&= 3(x-1)+0y+6(z-1)\\
&= 3x+6z-9
\end{aligned}$$

Also from exercise 19 we have $Hf(\mathbf{a})$, so $\mathbf{h}^T Hf(\mathbf{a})\mathbf{h}$ is the matrix product

$$\begin{aligned} & \mathbf{h}^T Hf(\mathbf{a})\mathbf{h} \\ = & \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \\ = & 6h_1^2 + 4h_1h_2 - 4h_2h_3 + 12h_3^2 \\ = & 6(x-1)^2 + 4(x-1)y - 4y(z-1) + 12(z-1)^2 \\ = & 6x^2 + 12z^2 + 4xy - 4yz - 12x - 24z + 18 \end{aligned}$$

Therefore, for $\mathbf{a} = (1, 0, 1)$, we can evaluate the Taylor polynomial

$$\begin{aligned} & p_2(x, y, z) \\ = & f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + \tfrac{1}{2}\mathbf{h}^T Hf(\mathbf{a})\mathbf{h} \\ = & 3 + (3x + 6z - 9) \\ & + \tfrac{1}{2}(6x^2 + 12z^2 + 4xy - 4yz - 12x - 24z + 18) \\ = & 3 - 3x - 6z + 3x^2 + 6z^2 + 2xy - 2yz \end{aligned}$$

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