Section 5.3 selected answers

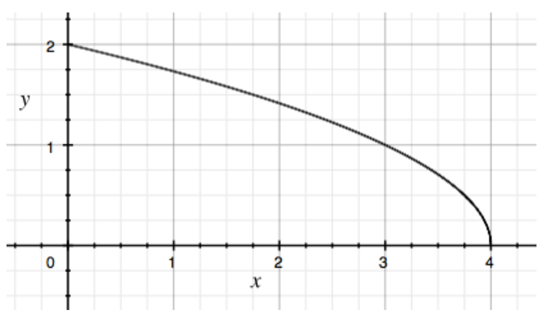
Math 131 Multivariate Calculus

D Joyce, Spring 2014

Exercises 3–6, 15, 17.

In exercises 3–6, the difficult part is determining the limits of integration when the order of integration is changed. In exercises 15 and 17, changing the order of integration is necessary to evaluate the integral.

4. The given integral is $\displaystyle\int_{y=0}^{2}\int_{x=0}^{4-y^2} x\,dx\,dy$, so the region D of integration is to the right of the y-axis, above the x-axis, and below the parabola $x = 4-y^2$.

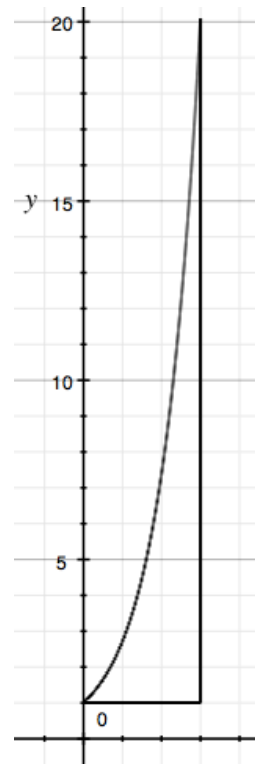


Note that D is bounded on the left by $x = 0$, and the rightmost point in R is at $(x, y) = (4, 0)$. The parabola $x = 4-y^2$ is also described by the equation $y = \sqrt{4-x}$; just solve the equation for y in terms of x. So for a given value of x between 0 and 4, y can vary from 0 to $\sqrt{4-x}$. We now have the limits of integration.

$$\int_{x=0}^{4}\int_{y=0}^{\sqrt{4-x}} x\,dx\,dy.$$

6. The given integral is $\displaystyle\int_{x=0}^{3}\int_{y=1}^{e^x} 2\,dy\,dx$, so the region D of integration is described by the inequalities

$$0 \le x \le 3, \qquad 1 \le y \le e^x.$$



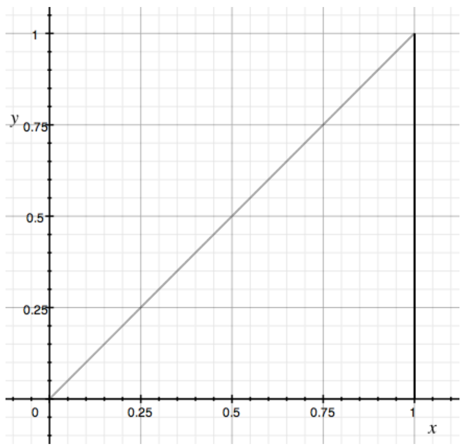
Note that the smallest value of y in R is 1, while the largest value of y in R is e^3. If you solve the equation $y = e^x$ for x in terms of y, you get $x = \ln y$. So for a given value of y between 1 and e^3, the smallest value of x is $\ln y$, while the largest value of x is 3. That gives this double integral:

$$\int_{y=1}^{e^3}\int_{x=\ln y}^{3} 2\,dx\,dy$$

15. Evaluate the iterated integral

$$\int_{0}^{1}\int_{y}^{1} x^2 \sin xy\,dx\,dy.$$

This can be solved without changing the limits of integration, but it would involve two applications of integration by parts. Let's try changing the limits of integration instead.



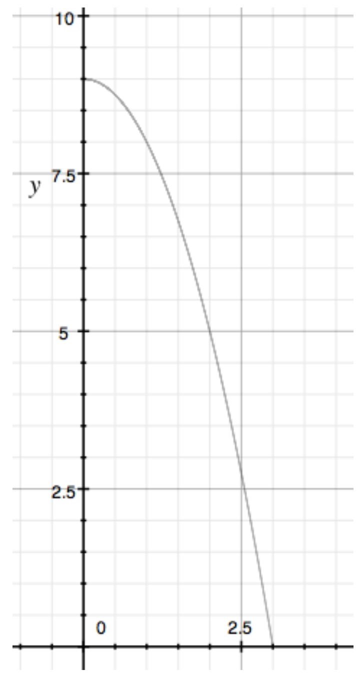
The region R is a triangle with vertices at $(0,0)$, $(1,1)$, and $(1,0)$. That means x can vary from 0 to 1, and for a given x in that interval, y can vary from 0 to x. That gives the integral

$$\begin{aligned}\int_0^1\int_0^x x^2\sin xy\,dy\,dx &= \int_0^1\left(-x\cos xy\,\Big|_{y=0}^{x}\right)dx\\ &= \int_0^1(-x\cos x^2+x)\,dx\\ &= -\tfrac{1}{2}\sin x^2+\tfrac{1}{2}x^2\,\Big|_0^1\\ &= -\tfrac{1}{2}\sin 1+\tfrac{1}{2}\end{aligned}$$

17. Evaluate the iterated integral

$$\int_0^3\int_0^{9-x^2}\frac{xe^{3y}}{9-y}\,dy\,dx.$$

Again, let's change the limits of integration. Solve $y=9-x^2$ for x to get $x=\sqrt{9-y}$. The least value of y in the region is $y=0$ while the greatest value of y is 9 (and that happens when $x=0$). So, the integral can be rewritten as



$$\begin{aligned}&\int_0^9\int_0^{\sqrt{9-y}}\frac{xe^{3y}}{9-y}\,dx\,dy\\ &= \int_0^9\frac{e^{3y}}{9-y}\int_0^{\sqrt{9-y}}x\,dx\,dy\\ &= \int_0^9\frac{e^{3y}}{9-y}\left(\tfrac{1}{2}x^2\,\Big|_0^{\sqrt{9-y}}\right)dy\\ &= \int_0^9\frac{e^{3y}}{9-y}\tfrac{1}{2}(9-y)\,dy\\ &= \tfrac{1}{2}\int_0^9 e^{3y}\,dy\\ &= \tfrac{1}{6}e^{3y}\,\Big|_0^9\\ &= \tfrac{1}{6}(e^{27}-1)\end{aligned}$$

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`