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Section 5.5 selected answers

Math 131 Multivariate Calculus

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Exercises from section 5.5: 1, 3, 9, 13, 17.

9. Evaluate the integral

$$\int_0^2 \int_{x/2}^{(x/2)+1} x^5(2y-x)e^{(2y-x)^2}\,dy\,dx$$

by making the substitution $u = x, v = 2y - x$.

One thing to do is to compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$. Note that when The inverse Jacobian is easy to find.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$$

So the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$.

Another thing to do is draw the domain of integration D in the (x,y)-plane, convert it to the corresponding domain of integration D^* in the (u,v)-plane, and determine the limits of integration for u and v. The domain D is a parallelogram with vertices at $P_1 = (0,0)$, $P_2 = (2,1)$, $P_3 = (0,1)$, and $P_4 = (2,2)$. Since the transformation is linear, the new domain D^* is also a parallelogram, in fact, it's a square with vertices at $P_1^* = (0,0)$, $P_2^* = (2,0)$, $P_3^* = (0,2)$, and $P_4^* = (2,2)$. Therefore, the new integral is

$$\int_0^2 \int_0^2 u^5 v e^{v^2}\,\tfrac{1}{2}\,dv\,du.$$

To evaluate this integral, first pass some constants through the integral signs:

$$\tfrac{1}{2}\int_0^2 u^5 \int_0^2 v e^{v^2}\,dv\,du.$$

For the inner integral, you can either make a substitution $w = v^2$ or see by inspection that an antiderivative of ve^{v^2} is $\frac{1}{2}e^{v^2}$.

$$\int_0^2 v e^{v^2}\,dv = \tfrac{1}{2}e^{v^2}\Big|_{v=0}^{2} = \tfrac{1}{2}(e^4 - 1)$$

Substitute that into the outer integral to get

$$\tfrac{1}{4}(e^4-1)\int_0^2 u^5\,du = \tfrac{8}{3}(e^4-1).$$

13. Transform to polar coordinates

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3\,dy\,dx.$$

The domain of this integral is the interior of the whole unit circle. (So you know the answer is 3 times the area of the unit circle, that is, 3π.)

Since the domain is the entire circle, θ has a range of 2π. You can make it either $0 \le \theta \le 2\pi$ or $-\pi \le \theta \le \pi$ as you like. Let's take the first one. The variable r will range from 0 to 1. That gives the integral in polar coordinates as

$$\int_0^{2\pi} \int_0^1 3r\,dr\,d\theta$$

where $dx\,dy$ is replaced by $r\,dr\,d\theta$ because the Jacobian equals r. The resulting integral is easy to evaluate.

17. Transform to polar coordinates

$$\int_0^3 \int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}}.$$

The domain of integration is a triangle with vertices at $(0,0)$, $(3,0)$, and $(3,3)$. Next, determine ranges for θ and r. The first one, θ ranges from 0 to $\pi/4$. For a given θ in that range, the range of r has to be determined. It will range from 0 to the distance from the origin to the point at the intersection of the θ-ray and the vertical line $x = 3$. All that has to be done to determine that distance is to convert the equation of that line to polar coordinates, then solve for r in terms of θ. The equation $x = 3$ converts to $r\cos\theta = 3$; therefore, $r = \frac{3}{\cos\theta}$. Thus, the integral in polar coordinates is

$$\int_0^{\pi/4} \int_0^{3/(\cos\theta)} \frac{1}{r}\, r\, dr\, d\theta.$$

The r's cancel, so the integral is easy to evaluate.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`