Section 6.1 selected answers

Math 131 Multivariate Calculus

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Exercises from section 6.1: 1–3, 9, 16, 17, 34

1. Let $f(x,y) = x + 2y$. Evaluate the scalar line integral $\int_{\mathbf{x}} f\,ds$ over the given path $\mathbf{x}$.

a. $\mathbf{x}(t) = (2-3t, 4t-1)$, $0 \leq t \leq 2$.

Since $\int_{\mathbf{x}} f\,ds = \int_a^b f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt$, we first need to compute the velocity $\mathbf{x}'$, then the speed $\|\mathbf{x}'\|$. The velocity is $\mathbf{x}' = (-3, 4)$, while the speed is $\|\mathbf{x}'\| = \sqrt{9+16} = 5$. Therefore, the line integral is

$$\begin{aligned} \int_{\mathbf{x}} f\,ds &= \int_a^b f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt \\ &= \int_0^2 (x+2y)5\,dt \\ &= 5\int_0^2 \big((2-3t) + 2(4t-1)\big)\,dt \\ &= 5\int_0^2 5t\,dt = 50 \end{aligned}$$

b. $\mathbf{x}(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$.

The velocity is $\mathbf{x}' = (-\sin t, \cos t)$, while the speed is $\|\mathbf{x}'\| = \sqrt{\sin^2 t + \cos^2 t} = 1$. Therefore, the line integral is

$$\begin{aligned} \int_{\mathbf{x}} f\,ds &= \int_a^b f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt \\ &= \int_0^\pi (x+2y)1\,dt \\ &= \int_0^\pi (\cos t + 2\sin t)\,dt \\ &= \sin t - 2\cos t\Big|_0^\pi = 4 \end{aligned}$$

2. Calculate the line integral $\int_{\mathbf{x}} f\,ds$ where $f(x,y,z) = xyz$, $\mathbf{x}(t) = (t, 2t, 3t)$, $0 \leq t \leq 2$.

The velocity is $\mathbf{x}' = (1,2,3)$, so the speed is $\|\mathbf{x}'\| = \sqrt{1+4+9} = \sqrt{14}$.

$$\begin{aligned} \int_{\mathbf{x}} f\,ds &= \int_a^b f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt \\ &= \int_0^2 xyz\sqrt{14}\,dt \\ &= \int_0^2 6t^3\sqrt{14}\,dt \\ &= \frac{6\sqrt{14}}{4}t^4\Big|_0^2 = 24\sqrt{14} \end{aligned}$$

3. Evaluate $\displaystyle\int_{\mathbf{x}} \frac{x+z}{y+z}\,ds$ over the path

$$\mathbf{x}(t) = (t, t, t^{3/2}) \text{ for } 1 \leq t \leq 3.$$

$$\begin{aligned} \mathbf{x}' &= (1, 1, \tfrac{3}{2}t^{1/2}) \\ \|\mathbf{x}'\| &= \sqrt{1+1+\tfrac{9}{4}t} = \tfrac{1}{2}\sqrt{8+9t} \\ \int_{\mathbf{x}} \frac{x+z}{y+z}\,ds &= \int_1^3 \frac{x+z}{y+z}\,\tfrac{1}{2}\sqrt{8+9t}\,dt \\ &= \tfrac{1}{2}\int_1^3 \frac{t+\frac{3}{2}t^{1/2}}{t+\frac{3}{2}t^{1/2}}\sqrt{8+9t}\,dt \\ &= \tfrac{1}{2}\int_1^3 \sqrt{8+9t}\,dt \\ &= \tfrac{1}{27}(8+9t)^{3/2}\Big|_1^3 \\ &= \tfrac{1}{27}\big(35^{3/2} - 17^{3/2}\big) \end{aligned}$$

9. Evaluate the vector integral $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$ where $\mathbf{F} = ((y+2), x)$ and $\mathbf{x}(t) = (\sin t, -\cos t)$ for $0 \leq t \leq$

$\pi/2$.

$$\begin{aligned}
&\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} \\
=\;& \int_0^{\pi/2} \left((y+2)\frac{dx}{dt} + x\frac{dy}{dt}\right)\,dt \\
=\;& \int_0^{\pi/2} \left((-\cos t+2)(\cos t) + (\sin t)(\sin t)\right)\,dt \\
=\;& \int_0^{\pi/2} (2\cos t - \cos^2 t + \sin^2 t)\,dt \\
=\;& 2\sin t - \frac{\sin 2t}{2}\bigg|_0^{\pi/2} = 2
\end{aligned}$$

16. Evaluate the vector integral $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$ where $\mathbf{F} = (y\cos z, x\sin z + xy\sin z^2)$ and $\mathbf{x}(t) = (t, t^2, t^3)$ for $0 \le t \le 1$.

$$\begin{aligned}
&\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} \\
=\;& \int_0^1 \left((y\cos z)\frac{dx}{dt} + (x\sin z)\frac{dy}{dt} + (xy\sin z^2)\frac{dz}{dt}\right)dt \\
=\;& \int_0^1 \left((y\cos z)(1) + (x\sin z)2t + (xy\sin z^2)3t^2\right)dt \\
=\;& \int_0^1 \left(t^2\cos t^3 + 2t^2\sin t^3 + 3t^5\sin t^6\right)dt \\
=\;& \frac{\sin t^3}{3} - \frac{2\cos t^3}{3} - \frac{\cos t^6}{2}\bigg|_0^1 \\
=\;& \frac{7 - 7\cos 1 + 2\sin 1}{6}
\end{aligned}$$

17. Determine the value of $\int_{\mathbf{x}} x\,dy - y\,dx$ over the path $\mathbf{x}(t) = (\cos 3t, \sin 3t)$ for $0 \le t \le \pi$.

$$\begin{aligned}
&\int_{\mathbf{x}} x\,dy - y\,dx \\
=\;& \int_0^{\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt \\
=\;& \int_0^{\pi} \left((\cos 3t)(3\cos 3t) - (\sin 3t)(-3\sin 3t)\right)dt \\
=\;& \int_0^{\pi} 3\,dt = 3\pi
\end{aligned}$$

34. Tom Sawyer is whitewashing a picket fence. The base of the fenceposts are arranged in the (x,y)-plane as the quarter circle $x^2 + y^2 = 25$ for $x, y \ge 0$, and the height of the fencepost at point (x,y) is given by $h(x,y) = 10 - x - y$. Use a scalar line integral to find the area of one side of the fence.

The line integral is $\int_{\mathbf{x}}(10 - x - y)\,ds$ where $\mathbf{x}$ is any path that describes that quarter circle. Let's take $\mathbf{x}(t) = (5\cos t, 5\sin t)$ for $0 \le t \le \pi/2$. For this path, the velocity is $\mathbf{x}'(t) = (-5\sin t, 5\cos t)$, so the speed is $\|\mathbf{x}'(t)\| = 5$. Then

$$\begin{aligned}
&\int_{\mathbf{x}} (10 - x - y)\,ds \\
=\;& \int_0^{\pi/2} (10 - x - y)\,\|\mathbf{x}'(t)\|\,dt \\
=\;& 5\int_0^{\pi/2} (10 - 5\cos t - 5\sin t)\,dt \\
=\;& 25(\pi - 2)
\end{aligned}$$

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