Section 6.2 selected answers

Math 131 Multivariate Calculus
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Exercises from section 6.2: 1, 2, 3, 7, 9, 15, 17.

1. Verify Green's theorem

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D (N_x - M_y)\,dA$$

when $\mathbf{F} = (M, N) = (-x^2y, xy^2)$, and D is the disk $x^2 + y^2 \leq 4$.

Let's first evaluate the line integral. Parameterize the circle $x^2 + y^2 = 4$ by $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$. Then $\mathbf{x}'(t) = (-2\sin t, 2\cos t)$, so $\|\mathbf{x}'(t)\| = 2$. Therefore,

$$\begin{aligned}
&\oint_{\partial D} M\,dx + N\,dy \\
&= \int_0^{2\pi} (Mx' + Ny')\,dt \\
&= \int_0^{2\pi} \big(-x^2y(-2\sin t) + xy^2(2\cos t)\big)\,dt \\
&= \int_0^{2\pi} 32\sin^2 t\cos^2 t\,dt
\end{aligned}$$

There are various ways to evaluate this integral, but double angle formulas are the easiest. Anyway, the value turns out to be 8π.

Next, evaluate the double integral. It's most easily evaluated using polar coordinates.

$$\begin{aligned}
\iint_D (N_x - M_y)\,dA &= \iint_D (y^2 - (-x^2))\,dA \\
&= \int_0^{2\pi}\int_0^2 r^2\,r\,dr\,d\theta \\
&= \int_0^{2\pi} 4\,d\theta = 8\pi
\end{aligned}$$

2. Verify Green's theorem when $\mathbf{F} = (M, N) = (x^2 - y, x + y^2)$, and D is the rectangle bounded by $x = 0$, $x = 2$, $y = 0$, and $y = 1$.

First, evaluate the line integral. The boundary ∂D is made of four line segments. There are various ways you can parametrize these four lines; here's one way.

$$\begin{array}{llll}
\mathbf{x}_1: & x = t, y = 0, & 0 \leq t \leq 2 & x' = 1, y' = 0 \\
\mathbf{x}_2: & x = 2, y = t, & 0 \leq t \leq 1 & x' = 0, y' = 1 \\
\mathbf{x}_3: & x = -t, y = 1, & -2 \leq t \leq 0 & x' = -1, y' = 0 \\
\mathbf{x}_2: & x = 0, y = -t, & -1 \leq t \leq 0 & x' = 0, y' = -1
\end{array}$$

Those parameterizations give these four integrals.

$$\int_0^2 \big((x^2 - y)x' + (x + y^2)y'\big)\,dt = \int_0^2 t^2\,dt = \tfrac{8}{3}$$

$$\int_0^1 \big((x^2 - y)x' + (x + y^2)y'\big)\,dt = \int_0^1 (4 + t^2)\,dt = \tfrac{4}{3}$$

$$\int_{-2}^0 \big((x^2 - y)x' + (x + y^2)y'\big)\,dt = \int_{-2}^0 (1 - t^2)\,dt = \tfrac{2}{3}$$

$$\int_{-1}^0 \big((x^2 - y)x' + (x + y^2)y'\big)\,dt = \int_{-1}^0 -t^2\,dt = \tfrac{1}{3}$$

The line integral is the sum of these four integrals, and that's 4.

Next, evaluate the double integral. Since

$$\iint_D (N_x - M_y)\,dA = \iint_D (1 - (-1))\,dA,$$

this is just 2 times the area of D. Since D is a rectangle of area 2, that gives 4.

9. Evaluate the line integral

$$\oint_C (x^2 - y^2)\,dx + (x^2 + y^2)\,dy$$

where C is the boundary of the unit square oriented *clockwise*. Use any method.

Let's see if we can convert it to a double integral. First, we can correct the orientation by negating the integral. We'll have

$$-\oint_{\partial D} (x^2 - y^2)\,dx + (x^2 + y^2)\,dy$$

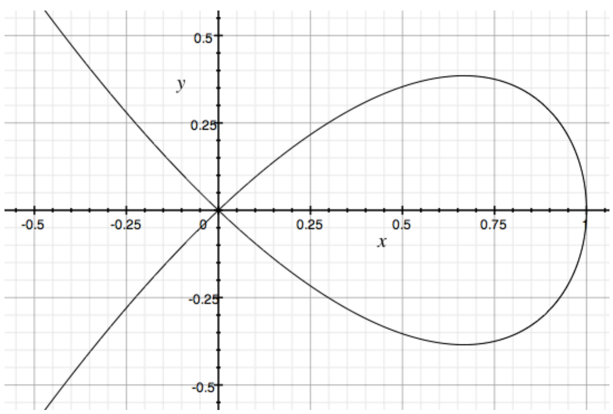
where D is the unit square. Green's theorem says that equals

$$\begin{aligned} & - \iint_D \left(\frac{d}{dx}(x^2+y^2) - \frac{d}{dy}(x^2-y^2) \right) dA \\ = & - \iint_D (2x+2y)\, dA \\ = & -2 \int_0^1 \int_0^1 (x+y)\, dy\, dx = -2 \end{aligned}$$

15. (a) Sketch the curve given parametrically by

$$\mathbf{x}(t) = (1-t^2, t^3-t).$$

You'll notice that when t is equal to either 1 or -1 that $\mathbf{x} = (0,0)$. Between those values the curve loops around.



Note that the loop is traversed in the clockwise direction.

15. (b) Find the area inside the closed loop of the curve.

We can use Green's theorem to find the area D bounded by a curve ∂D by evaluating a certain line integral over that curve as explained in the text.

$$\text{Area}(D) = \iint_D 1\, dx\, dy = \tfrac{1}{2} \oint_{\partial D} -y\, dx + x\, dy$$

In this exercise, we have a parameterization of ∂D in part **a**, but to get the orientation on the boundary right, we'll have to make have to make t vary backwards from 1 to -1.

$$\tfrac{1}{2} \oint_{\partial D} -y\, dx + x\, dy = \tfrac{1}{2} \int_1^{-1} (-yx' + xy')\, dt$$

The integrand is

$$\begin{aligned} & -yx' + xy' \\ = & -(t^3-t)(-2t) + (1-t^2)(3t^2-1) \\ = & -t^4 + 4t^2 - 1 \end{aligned}$$

So the integral is equal to

$$\tfrac{1}{2}(-\tfrac{1}{5}t^5 + \tfrac{4}{3}t^3 - t)\Big|_1^{-1}$$

which evaluates as $\frac{8}{15}$.

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`http://math.clarku.edu/~djoyce/ma131/`