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Section 6.3 selected answers

Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises from section 6.3: 3–6.

In each exercise, determine whether the given vector field is conservative. If it is, find a scalar potential function for it.

3. $\mathbf{F} = e^{x+y}\,\mathbf{i} + e^{xy}\,\mathbf{j}$.

Probably the easiest way to determine if a vector field is conservative is to see if its curl is 0. In the two-dimensional case, the curl of $\mathbf{F} = (M, N)$ is $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$. So, you can either show this curl is 0, or you can show that the two partial derivatives, N_x and M_y, are equal.

$$\begin{aligned}\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \frac{\partial}{\partial x}e^{xy} - \frac{\partial}{\partial y}e^{x+y}\\ &= ye^{xy} - e^{x+y}\end{aligned}$$

Since this curl is not 0, the vector field is not conservative.

4. $\mathbf{F} = 2x\sin y\,\mathbf{i} + x^2\cos y\,\mathbf{j}$.

$$\begin{aligned}\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \frac{\partial}{\partial x}x^2\cos y - \frac{\partial}{\partial y}2x\sin y\\ &= 2x\cos y - 2x\cos y = 0\end{aligned}$$

Since this curl is 0, the vector field is conservative.

Now we need to find a scalar function $f(x, y)$ such that $\nabla f = \mathbf{F}$. That is, $f_x = M$ and $f_y = N$. In this exercise, we need

$$\begin{aligned}f_x &= 2x\sin y\\ f_y &= x^2\cos y\end{aligned}$$

Integrate the first equation.

$$\int 2x\sin y\,dx = x^2\sin y + C(y)$$

where $C(y)$ does not depend on x but can depend on y. But $\frac{\partial}{\partial y}x^2\sin y = x^2\cos y$, so we can take $C(y)$ to be 0. Thus, a potential function for $\mathbf{F}$ is $f(x, y) = x^2\sin y$.

5. $\mathbf{F} = \left(3x^2\cos y + \dfrac{y}{1+x^2y^2}, x^3\sin y + \dfrac{x}{1+x^2y^2}\right)$.

$$\begin{aligned}&\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\\ &= \left(3x^2\sin y + \frac{1(1+x^2y^2) - x(2xy^2)}{(1+x^2y^2)^2}\right) - \\ &\quad\left(-3x^2\sin y + \frac{1(1+x^2y^2) - y(2x^2y)}{(1+x^2y^2)^2}\right)\\ &= 6x^2\sin y\end{aligned}$$

Since this is not 0, $\mathbf{F}$ is not conservative.

6. $\mathbf{F} = \dfrac{xy^2}{(1+x^2)^2}\,\mathbf{i} + \dfrac{x^2y}{1+x^2}\,\mathbf{j}$.

$$\begin{aligned}\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \frac{2xy(1+x^2) - x^2y(2x)}{(1+x^2)^2} - \frac{2xy}{(1+x^2)^2}\\ &= 0\end{aligned}$$

So $\mathbf{F}$ is conservative. Its potential function f satisfies

$$\begin{aligned}f_x &= \frac{xy^2}{(1+x^2)^2}\\ f_y &= \frac{x^2y}{1+x^2}\end{aligned}$$

It's probably easier to integrate the second equation, but let's do the first anyway. The substitution $u = 1 + x^2$, $du = 2x\,dx$ simplifies it.

$$\begin{aligned}\int \frac{xy^2}{(1+x^2)^2}\,dx &= \int \frac{y^2}{2u^2}\,du\\ &= -\frac{y^2}{2u} + C(y)\\ &= -\frac{y^2}{2(1+x^2)} + C(y)\end{aligned}$$

This time, the second equation is not satisfied with $C(y) = 0$, so we have to figure out what $C(y)$ is.

$$f_y = \frac{\partial}{\partial y}\left(-\frac{y^2}{2(1+x^2)} + C(y)\right) = -\frac{y}{1+x^2} + C'(y)$$

Therefore, we need

$$\frac{x^2 y}{1+x^2} = -\frac{y}{1+x^2} + C'(y).$$

That equation simplifies to

$$y = C'(y),$$

so $C(y) = \frac{1}{2}y^2$ will do. Thus,

$$f(x,y) = -\frac{y^2}{2(1+x^2)} + \frac{y^2}{2} = \frac{x^2y^2}{2(1+x^2)}$$

is a potential function for $\mathbf{F}$.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`