Surface integrals
Math 131 Multivariate Calculus
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The area differential of a surface, and a double integral for the area of the surface. Recall that we're using $\mathbf{X}(s,t)$ to describe a paramaterization of a surface S in 3-space. Also we have the tangent vectors $\mathbf{T}_s$ and $\mathbf{T}_t$ at each point in the surface defined by

$$\mathbf{T}_s = \mathbf{X}_s = \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right) \quad \text{and} \quad \mathbf{T}_t = \mathbf{X}_t = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$$

and the normal vector $\mathbf{N}$ defined in terms of them $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$.

We can use $\mathbf{T}_s$, $\mathbf{T}_t$, and $\mathbf{N}$ to define a surface area differential dS of a surface S.

Let S be a surface parameterized by $\mathbf{X} : D \to \mathbf{R}^3$. A point $(s_0, t_0) \in D$, is mapped to $\mathbf{X}(s_0, t_0) \in \mathbf{R}^3$. An infinitesimal $dx \times dt$ parallelogram at $(s_0, t_0) \in D$ has area $dx\,dt$. It's mapped to an infinitesimal $\mathbf{T}_s(s_0,t_0)ds \times \mathbf{T}_t(s_0,t_0)$ rectangle with area $\|\mathbf{T}_s \times \mathbf{T}_t\|\,ds\,dt$, which equals $\|\mathbf{N}\|\,ds\,dt$. We'll call this infinitesimal parallelogram the *surface area differential*, denoted dS. Thus,

$$dS = \|\mathbf{N}\|\,ds\,dt = \|\mathbf{T}_s \times \mathbf{T}_t\|\,ds\,dt,$$

By summing these surface area differentials dS over the whole surface, we'll get the area of the surface

$$\text{Area of } S = \iint_D dS,$$

where D is the domain of the parametrization $\mathbf{X}$ describing the surface.

We can find $\mathbf{N}$, the normal vector, in terms of the components of $\mathbf{X}$ as follows.

$$\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\
&= \left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right)\mathbf{i} - \left(\frac{\partial x}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial z}{\partial s}\right)\mathbf{j} - \left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)\mathbf{k} \\
&= \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}
\end{aligned}$$

where the last line uses the same notation that we used for Jacobians. Therefore,

$$\|\mathbf{N}\| = \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}.$$

That gives us a more detailed expression for the surface area differential

$$dS = \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\,ds\,dt.$$

Graphs $z = f(x, y)$ of functions of two variables. One of the most common applications of surfaces in $\mathbf{R}^3$ is as graphs $z = f(x, y)$ of functions of two variables. These can easily be paramaterized by identifying s with x and t with y. Then $z = f(x, y)$. That is, $\mathbf{X}(s, t) = (s, t, f(s, t))$. Then

$$\mathbf{T}_s = \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right) = \left(1, 0, \frac{\partial f}{\partial s}\right) \quad \text{and} \quad \mathbf{T}_t = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = \left(0, 1, \frac{\partial f}{\partial t}\right).$$

Therefore,

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \left(-\frac{\partial f}{\partial s}, -\frac{\partial f}{\partial t}, 1\right),$$

which we can also write in terms of x and y as

$$\mathbf{N} = (-f_x, -f_y, 1).$$

So, in this case, the surface area differential is

$$dS = \|\mathbf{N}\|\, ds\, dt = \|\mathbf{N}\|\, dx\, dy = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy,$$

and an integral giving the surface area of the surface $z = f(x, y)$ over the domain D of f is

$$\text{Area } = \iint_D dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

Scalar surface integrals. Now that we have the surface differential dS, we can use it for more than just the area of the surface. The area is the integral of 1:

$$\text{Area } = \iint_D 1\, dS.$$

We can replace 1 by a function $f(x, y, z)$ to integrate f.

Here, $f(x, y, z)$ is a scalar-valued function $\mathbf{R}^3 \to \mathbf{R}$ whose domain includes the surface S. We can think of $f(x, y, z)$ being the weight, or density, at (x, y, z) on the surface. If f is constantly 1, then every point weighs the same, and the surface integral $\iint_D f\, dS$ just gives the area of S. But when f isn't constantly 1, then different points carry different weights. Thus, we make our definition of scalar surface integrals.

Definition 1. Let S be a surface in $\mathbf{R}^3$ parametrized by $\mathbf{X} : D \to \mathbf{R}^3$ where the domain D of the parameterization is a bounded set in $\mathbf{R}^2$ and the parametrization $\mathbf{X}$ is smooth (that is, C^1). We define the *scalar surface integral* of f as

$$\begin{aligned}
\iint_{\mathbf{X}} f\, dS &= \iint_D f(\mathbf{X}(s, t))\, \|N(s, t)\|\, ds\, dt \\
&= \iint_D f(\mathbf{X}(s, t))\, \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt \\
&= \iint_D f(\mathbf{X}(s, t)) \sqrt{\left(\frac{\partial(y, z)}{\partial(s, t)}\right)^2 + \left(\frac{\partial(x, z)}{\partial(s, t)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(s, t)}\right)^2}\, ds\, dt.
\end{aligned}$$

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