Math 131 Multivariate Calculus

First Sample Test Answers

1. [12] On functions of several variables.

a. Give an example of a function $\mathbf{f}: \mathbf{R}^3 \to \mathbf{R}^2$.

A typical such function would be

$$f(x,y,z) = (2x+3y, 4y+5z).$$

b. Give an example of a scalar-valued function whose domain is the set

$$\{(x,y) \in \mathbf{R}^2 | (x,y) \neq (0,0)\}.$$

Probably the simplest such function is

$$f(x,y) = \frac{1}{x^2+y^2}.$$

A common error for this problem is to give a function which is not defined when either $x=0$ or $y=0$. For example, the function $f(x,y) = \dfrac{1}{xy}$ is not defined on either the x-axis or the y-axis, and its domain is the set $\{(x,y) \in \mathbf{R}^2 | x \neq 0 \text{ and } y \neq 0\}$.

c. Determine the domain of the function $f(x,y) = \dfrac{\sqrt{x-10}}{y+5}$.

In order for f to be defined, x has to be at least 10 and y cannot equal 5. You can answer this in set theory notation as

$$\{(x,y) \in \mathbf{R}^2 \,|\, x \geq 10 \text{ and } y \neq 5\}.$$

2. [10] Consider the function $z = f(x,y) = y - x^2$. Draw either the level curve or the contour curve for f at height $c = 2$. (Draw one or the other. It's your choice.)

The equation $y - x^2 = 2$ is a parabola opening upward with its vertex on the y-axis at $y = 2$. This parabola in the (x,y)-plane is the level curve at height $c = 2$. The corresponding contour curve is the parabola $\{(x,y,2) \,|\, y - x^2 = 2\}$ which lies in the plane $z = 2$.

3. [20] On limits. Evaluate the limit, or explain why it fails to exist.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

On the line $x = 0$,

$$\lim_{y\to 0} \frac{x^2}{x^2+y^2} = \lim_{y\to 0} 0 = 0,$$

but on the line $y = 0$,

$$\lim_{x\to 0} \frac{x^2}{x^2+y^2} = \lim_{y\to 0} 1 = 1.$$

Since the value depends on the direction that (x,y) approaches $(0,0)$, therefore the limit doesn't exist.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x+2) = 2.$$

4. [32] On derivatives.

a. [5] Find $\dfrac{\partial f}{\partial y}$ if $f(x,y) = (x+e^x)\sin y$.

$$\frac{\partial f}{\partial y} = (x+e^x)\cos y$$

b. [5] Find $\dfrac{\partial^2 f}{\partial x \partial y}$ for the function f given in part a.

Differentiate the answer to part a with respect to y to get

$$\frac{\partial^2 f}{\partial x \partial y} = (1+e^x)\cos y$$

c. [10] Compute the gradient ∇f if $f(x,y,z) = (2x+3y+4z)^2$.

The gradient is the vector with the partial derivatives as its components.

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \big(4(2x+3y+4z), 6(2x+3y+4z), 8(2x+3y+4z)\big)\end{aligned}$$

There are other ways you can write this gradient, such as

$$\begin{aligned}\nabla f &= (8x+12y+16z, 12x+18y+24z, 16x+24y+31z) \\ &= (8x+12y+16z)\mathbf{i} \\ &\quad +(12x+18y+24z)\mathbf{j} \\ &\quad +(16x+24y+31z)\mathbf{k} \\ &= (2x+3y+4z)(4,6,8)\end{aligned}$$

d. [12] Find the derivative $D\mathbf{f}$ if $\mathbf{f}(x,y) = (2x+3y, xy, \cos x)$.

This derivative is a 3×2 matrix of partial derivatives.

$$D\mathbf{f} = \begin{bmatrix} 2 & 3 \\ y & x \\ -\sin x & 0 \end{bmatrix}$$

5. [17] On the chain rule. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^3$ has the derivative

$$D\mathbf{f}(x,y) = \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ y\cos xy & x\cos xy \end{bmatrix}$$

and $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ has the derivative $D\mathbf{x}(t) = \begin{bmatrix} 3t^2 \\ e^t \end{bmatrix}$.

a. [5] The derivative $D(\mathbf{f} \circ \mathbf{x})(t)$ is a matrix. What size is that matrix?

It's a 3×1 matrix.

b. [12] Find the derivative $D(\mathbf{f} \circ \mathbf{x})(t)$.

$$\begin{aligned} D(\mathbf{f} \circ \mathbf{x})(t) &= \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ y\cos xy & x\cos xy \end{bmatrix} \begin{bmatrix} 3t^2 \\ e^t \end{bmatrix} \\ &= \begin{bmatrix} 6xyt^2 + x^2e^t \\ 3t^2 + 2ye^t \\ 3yt^2\cos xy + xe^t\cos xy \end{bmatrix} \end{aligned}$$

Since you don't know exactly what $\mathbf{x} = (x,y)$ is in terms of t, leave the answer in terms of x, y, and t. (Actually, you could integrate to see that $x(t) = t^3 + C_1$ and $y(t) = e^t + C_2$ where C_1 and C_2 are unknown constants.)

6. [10] On differentiability. Consider the function whose graph $z = f(x,y)$ is displayed below. (See the textbook, page 112 figure 2.50). It is defined in terms of absolute values by

$$f(x,y) = |(|x| - |y|)| - |x| - |y|.$$

As you can see from the graph, it's partial derivatives evaluated at the origin are both 0, that is, $f_x(0,0) = f_y(0,0) = 0$. Explain why this function is not differentiable at the origin even though it's partial derivatives are both 0.

The surface has no tangent plane at the origin. There are problems in every direction except in the directions of the two axes. For instance, along the line $y = x$, you see valleys on either side of the the origin. On the negative half of this line, the slope is a positive constant, but on the positive half, it's a negative constant. In other words, the intersection of this surface with the plane $x = y$ is a curve with a corner at the origin.