Math 131 Multivariate Calculus
Second Sample Test Answers

1. [15] Suppose that a scalar-valued function $f : \mathbf{R}^3 \to \mathbf{R}$ has the gradient

$$\nabla f(x,y,z) = \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right).$$

Compute the directional derivative $D_{\mathbf{u}}(\mathbf{a})$ in the direction $\mathbf{u} = \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$ at the point $\mathbf{a} = (1, 2, -1)$.

First of all, the gradient at $\mathbf{a}$ is

$$\nabla f(1,2,-1) = \left(\tfrac{2}{6}, \tfrac{4}{6}, \tfrac{-2}{6}\right) = \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{-1}{3}\right).$$

Then, the directional derivative is that gradient dot the direction

$$\begin{aligned} \nabla f(x,y,z) &= \nabla f(\mathbf{a}) \cdot \mathbf{u} \\ &= \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{-1}{3}\right) \cdot \left(\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7}\right) \\ &= \tfrac{10}{21} \end{aligned}$$

2. [20] Calculate the velocity, speed, acceleration, and unit tangent vector of the path $\mathbf{x}(t) = (\cos t, \sin t, e^t)$.

a. Velocity. $\mathbf{x}'(t) = (-\sin t, \cos t, e^t)$.

b. Speed.

$$\begin{aligned} \|\mathbf{x}'(t)\| &= \|(-\sin t, \cos t, e^t)\| \\ &= \sqrt{\sin^2 t + \cos^2 t + e^{2t}} = \sqrt{1 + e^{2t}} \end{aligned}$$

c. Acceleration. $\mathbf{x}''(t) = (-\cos t, -\sin t, e^t)$.

d. Unit tangent vector.

$$T(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \left(\frac{-\sin t}{\sqrt{1+e^{2t}}}, \frac{\cos t}{\sqrt{1+e^{2t}}}, \frac{e^t}{\sqrt{1+e^{2t}}} \right)$$

3. [15] Set up an integral that gives the length of the path $\mathbf{x}(t) = (t^2, 1, 2\ln t)$ for $1 \le t \le 5$. Do not evaluate the integral.

First find the velocity and the speed. $\mathbf{x}'(t) = (2t, 0, 2/t)$. $\|\mathbf{x}'(t)\| = \|(2t, 0, 2/t)\| = \sqrt{4t^2 + 4/t^2} = 2\sqrt{t^2 + t^{-2}}$. Then the length of the path is

$$\int_a^b \|\mathbf{x}'(t)\|\, dt = \int_1^5 2\sqrt{t^2 + t^{-2}}\, dt.$$

4. [20] Consider the function

$$f(x,y) = e^{-y}(x^2 - y^2).$$

Its first and second partial derivatives are

$$f_x = 2xe^{-y} \qquad f_y = -e^{-y}(x^2 + 2y - y^2)$$

$$f_{xx} = 2e^{-y} \qquad f_{xy} = -2xe^{-y} \qquad f_{yy} = e^{-y}(x^2+4y-y^2-2)$$

a. Determine the two critical points of f.

To find the critical points set the two first partial derivatives to 0 and solve the system of two equations simultaneously. From the equation $f_x = 2xe^{-y} = 0$, it follows that $x = 0$. From the equation $f_y = -e^{-y}(x^2 + 2y - y^2) = 0$ it follows that $2y - y^2 = 0$, so $y = 0, 2$. Thus, the two critical points are

$$(x, y) = (0, 0), (0, 2).$$

b. Identify the nature (max, min, saddle) of each critical point.

First, write down the general Hessian

$$\begin{aligned} Hf &= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & e^{-y}(x^2 + 4y - y^2 - 2) \end{bmatrix} \end{aligned}$$

Now consider the first critical point $(x, y) = (0, 0)$.

$$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

The first principal minor is upper left entry of that matrix, $d_1 = 2$, while the second is the determinant of that matrix, $d_2 = -4$. Since d_1 is positive but d_2 is negative, this critical point is a saddle point.

Next consider the second critical point $(x, y) = (0, 2)$.

$$Hf(0,2) = \begin{bmatrix} 2e^{-2} & 0 \\ 0 & 2e^{-2} \end{bmatrix}.$$

Therefore, $d_1 = 2e^{-2}$ is positive, and $d_2 = 4e^{-4}$ is also positive. Since both principle minors are positive, this critical point is a minimum.

5. [15] The vector-valued field $\mathbf{F}(x, y, z) = (y, x, -3)$ is a gradient field. Find a potential function $f : \mathbf{R}^3 \to \mathbf{R}$ for $\mathbf{F}$.

We're looking for a function f such that $\nabla f = \mathbf{F}$. So we need $f_x = y$, $f_y = x$, and $f_z = -3$. Well, a function whose partial derivative with respect to x equals y is $xy + C$ where C doesn't depend on x. But the partial derivative of $xy + C$ with respect to y should be x, therefore C doesn't depend on y either. Finally, we need the partial derivative of $xy + C$ with respect to z to be -3, so $C = -3z$ works. Thus, a potential function for $\mathbf{F}$ is $f(x, y, z) = xy - 3z$. (All the rest of the potential functions for $\mathbf{F}$ differ from this one by a constant.)

6. [15] Set up a double integral to compute the volume of a solid whose base is the plane region D in the (x, y)-plane bounded by the x-axis and the parabola $y = 4 - x^2$; and whose height at a point (x, y) in that region is given by $f(x, y) = \sin(x^2 + y^2)$. Be sure to sketch the region D. Do not evaluate the integral.

The parabola $y = 4 - x^2$ opens downward and interects the x-axis at $x = 2$ and $x = -2$. It also intesects the y-axis at $y = 4$. So, the region D between that parabola and the x-axis is bounded between the vertical lines $x = -2$ and $x = 2$. Thus, a double integral for the volume is

$$\int_{x=-2}^{2} \int_{y=0}^{4-x^2} \sin(x^2 + y^2)\, dy\, dx.$$

Alternatively, you could slice D by horizontal lines to get a different integral. You'll need to solve $y = 4 - x^2$ for x to get $x = \pm\sqrt{4 - y}$.

$$\int_{y=0}^{4} \int_{x=-\sqrt{4-y}}^{\sqrt{4-y}} \sin(x^2 + y^2)\, dx\, dy.$$