Math 217, Probability and Statistics

First Test Answers

Sep 2005

Scale. 88-100 A. 76-87 B. 55-75 C. Median 93.

Problem 1. [12] Give an example of a probability density function f for a nonuniform continuous random variable. (You get to choose which one; just make sure it's not a uniform one.)

The function f needs to be a nonnegative function whose integral $\int_{-\infty}^{\infty} f(x)\,dx$ equals 1. That is, the area under the curve $y = f(x)$ should be 1. An example of such a function is

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2. [12] A fair die is tossed 6 times. What is the probability that exactly three 5's appear?

This is a Bernoulli process. The probability of success is $p = \frac{1}{6}$. We want the probability of $k = 3$ successes in $n = 6$ trials. Since

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k},$$

therefore, the probability we're looking for is

$$b(6, \tfrac{1}{6}, 3) = \binom{6}{3} (\tfrac{1}{6})^3 (\tfrac{5}{6})^3.$$

Numerically, that's 0.0535837.

Problem 3. [16; 4 points each part] For each of the following functions $F(x)$, could F be a cumulative distribution function for a continuous random variable? You don't have explain your answer; just write "yes" or "no".

a. $F(x) = \begin{cases} 0 & \text{if} & x \leq 0 \\ x/2 & \text{if} & 0 < x < 2 \\ 1 & \text{if} & 2 \leq x \end{cases}$

Yes. It's a cumulative distribution function for the uniform random variable on $[0, 2]$.

b. $F(x) = \begin{cases} 0 & \text{if} & x \leq 2 \\ 1/8 & \text{if} & 2 < x < 10 \\ 0 & \text{if} & 10 \leq x \end{cases}$

No, since it's not an increasing function. But it is the density function for the uniform random variable on $[2, 10]$.

c. $F(x) = \begin{cases} 0 & \text{if} & x \leq 0 \\ 1 - e^{-x} & \text{if} & 0 < x \end{cases}$

Yes. This is the cumulative distribution function for an exponential distribution.

d. $F(x) = \begin{cases} 0 & \text{if} & x < 0 \\ 1/over2 & \text{if} & 0 \leq x < 1 \\ 1 & \text{if} & 1 \leq x \end{cases}$

No. This may be a cumulative distribution function, but not for a continuous random variable. If it were, then it's derivative would be density function f, and F would be the integral of f. But F is not differentiable since it's not continuous, and F is not the integral of anything.

In fact, it's the cumulative distribution function for a Bernoulli trial.

Problem 4. [12; 6 points each part] A dart is thrown at a circular dartboard of radius 10 inches. Assume that the location that the dart lands is a uniform continuous distribution, that is, the probability that it lands in a region of the dartboard is proportional to the area of that region.

(a). What is the probability that the dart falls within 5 inches of the center of the target?

The area of a circle of radius 5 in the center of the target is $\pi 5^2$. The area of the entire target is $\pi 10^2$. Therefore, the probability that a dart lands in the region described is $\dfrac{\pi 5^2}{\pi 10^2}$, and that equals $\frac{1}{4}$.

(b). What is the probability that the dart falls within 3 inches of the edge of the target?

If a dart falls within 3 inches of the edge of the target, it misses the circle in the center of the target of radius 7. That circle has probability $\dfrac{\pi 7^2}{\pi 10^2} = \frac{49}{100}$. Therefore, the probability it lands within 3 inches of the edge is $1 - \frac{49}{100} = \frac{51}{100}$.

Problem 5. [12] Two fair dice are tossed, one red and one green. What is the probability that they show the same number? Explain your answer in terms of outcomes in a sample space.

The sample space consists of 36 outcomes, each outcome being an ordered pair, the number on the red die, and the number on the green die. Six of these outcomes, namely

$(1,1), (2,2), \ldots, (6,6)$, are where the red and greed dice show the same number. Since this is uniform discrete probability, the probability that they show the same number is the quotient $\frac{6}{36}$, which equals $\frac{1}{6}$.

Problem 6. [12] Four cards are dealt from a standard deck of cards (4 suits, 13 cards in each suit). What is the probability that one of the four cards is in each suit (that is, no two of the four cards are in the same suit). Explain your reasoning.

There are essentially three ways to approach this problem. The first involves combinations and ignores the order that the four cards were dealt. For this approach, a hand consists of a subset of size 4 of the set of 52 cards. There are $\binom{52}{4}$ such subsets. For a hand to have one card in each of the suits, it must have one spade (out of 13 possible spades), one heart (out of 13 possible hearts), one diamond (out of 13 possible diamonds), and one club (out of 13 possible clubs). There are 13^4 such hands. Therefore, the probability that a hand has one card in each suit is $13^4/\binom{52}{4}$. Numerically, that's $\frac{28561}{270725} = 0.105498$.

The second approach involves partial permutations and keeps track of the order that the cards were dealt. For this approach, a hand consists of an ordered subset of four cards—the first card dealt, the second dealt, the third dealt, and the fourth dealt. So a hand is a 4-permutation from a set of size 52. There are $\frac{52!}{48!} = 52 \cdot 51 \cdot 50 \cdot 49$ of them. For this hand have one card in each suit, the first card may be any of the 52 cards, the second any of the 39 cards not in the same suit as the first, the third any of the 26 cards not in the same suit of either of the first two, and the fourth any of the 13 cards in the remaining suit. That makes $52 \cdot 39 \cdot 26 \cdot 13$ (which is $13^4 4!$) hands when the order that the cards were dealt is retained. Therefore, the probability that a hand has one card in each suit is $\dfrac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49}$. That simplifies to $\dfrac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49}$, which, again, equals 0.105498.

The third approach involves conditional probability, but it's very close to the second approach. The first card dealt may be any card. The probability that the second card dealt is in a different suit than the first card is $\frac{39}{51}$ since there are 39 cards in the other three suits, but 51 cards still in the deck. Given that the first two cards dealt are in different suits, the probability that the third card dealt is in a different suit than either of them is $\frac{26}{50}$ since there are 26 cards in the remaining two suits, but 50 cards still in the deck. Given that the first three cards dealt are in different suits, the probability that the fourth card dealt is in the last suit then is $\frac{13}{49}$ since there are 13 cards in the remaining suit, but 49 cards still in the deck. Multiplying these probabilities together gives the answer

$$\frac{39}{51}\,\frac{26}{50}\,\frac{13}{49} = 0.105498.$$

Problem 7. [12] Suppose a number X is chosen randomly and uniformly from the interval $[-10, 10]$. Determine $P(X^2 > 4)$.

Since X is a uniform random variable on the interval $[-10, 10]$, and that interval has length 20, therefore the probability that X lies in a subinterval is the length of that subinterval divided by 20.

Now $X^2 > 4$ will hold either when $X < -2$ or $2 < X$. The first condition, $X < -2$ holds when X lies in the interval from -10 to -2, and that interval has length 8, so $P(X < -2) = \frac{8}{20}$. The second condition, $2 < X$ holds when X lies in the interval from 2 to 10, and that interval also has length 8, so $P(2 < 10) = \frac{8}{20}$. Therefore,

$$\begin{aligned} P(X^2 > 4) &= P(X < -2) + P(2 < 10) \\ &= \tfrac{8}{20} + \tfrac{8}{20} = \tfrac{16}{20} = \tfrac{4}{5}. \end{aligned}$$

Problem 8. [12] Recall that a set Ω is said to be *partitioned* into subsets $A_1, A_2, \ldots, A_n$ if each element of Ω belongs to exactly one of the subsets A_i. Let $m : \Omega \to [0, 1]$ be a discrete probability distribution on Ω. Prove that if Ω is partitioned into the events $A_1, A_2, \ldots, A_n$ then

$$P(A_1) + P(A_2) + \cdots + P(A_n) = 1.$$

Base your proof on outcomes in the sample space Ω.

There are many ways you can put this proof into words, and at least a couple of different ways to approach the proof. Here's one proof.

Since no outcome in Ω lies in more than one subset $A_1, A_2, \ldots, A_n$, therefore the probability of their union equals the sum of their probabilities

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

But every outcome in Ω lies is some subset $A_1, A_2, \ldots, A_n$, therefore their union is the entire set Ω. Therefore,

$$P(\Omega) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Now, since $P(\Omega) = 1$, we may conclude that $P(A_1)+P(A_2)+\cdots+P(A_n) = 1$. Q.E.D.