You may refer to two sheets of notes on this test, the summary table of distributions below, the table for the normal distribution, and you may use a calculator. Show your work for credit. You may leave your answers as expressions such as \( \left( \frac{8}{4} \right)^{1/3} \frac{e^{1/3}}{\sqrt{2\pi}} \) if you like.

1. One of the cards of an ordinary deck of 52 cards is lost.
   a. Given that it is not a king, what is the probability that it’s an ace?
   b. Given that it’s either a heart or a diamond, what is the probability that it’s an ace.
   c. Is the event that the lost card is not a king independent of the event that the lost card is an ace?
   d. Is the event that the lost card is either a heart or a diamond independent of the event that the lost card is an ace?

2. Let \( X \) be a continuous random variable with cumulative distribution function

\[
F_X(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\sqrt{x} & \text{if } 0 \leq x \leq 1 \\
1 & \text{if } 1 \leq x
\end{cases}
\]

a. Determine the probability that \( \frac{1}{3} \leq X \leq \frac{2}{3} \).

b. Determine the density function \( f_X(x) \).

c. Determine the mean \( \mu_X \) of \( X \).

d. Determine the variance \( \sigma^2_X \) of \( X \).

3. Suppose that \( X \) is a random variable with moment generating function \( g(t) = (0.3 + 0.7e^t)^5 \). Note that the first and second derivatives of \( g \) are

\[
g'(t) = 3.5(0.3 + 0.7e^t)^4e^t, \quad \text{and} \quad g'' = 3.5(0.3 + 3.5e^t)(0.3 + 0.7e^t)^3e^t.
\]

a. Determine the mean \( \mu_X \) of \( X \).

b. Determine the variance \( \sigma^2_X \) of \( X \).

4. Ann and Peter sell cars at a local dealership. The number of cars which Ann sells during a week is assumed to have a Poisson distribution with an average of 5 cars/week, and the number of cars which Peter sells during a week is assumed to have a Poisson distribution with an average of 4 cars/week. Assume that the numbers of cars/week that Ann and Peter sell are independent.

a. What is the probability that Ann sells exactly 5 cars in a given week?

b. What is the probability that Peter sells no cars in a given week?

c. What is the probability that together they sell exactly 10 cars in a given week?
5. Suppose that (a) 1 out of 10,000 of women at age forty have breast cancer. Further, suppose that (b) 99% of the women who do have breast cancer will get a positive mammogram, and (c) 1% of the women who do not have breast cancer will also get a positive mammogram. (d) If a woman in this age group gets a positive mammogram, how likely is it that she actually has breast cancer?

   a-c. Let $E$ be the event that a randomly chosen women at age forty has breast cancer, and let $F$ be the event that a randomly chosen woman has a positive mammogram. Express each of the statements (a) through (c) in terms of probability or conditional probability.

   d. Given statements (a-c) are correct, determine the probability for (d). (Show your work. You can leave your answer as an expression involving numbers or you can evaluate it decimally using a calculator.)

6. A fair die is rolled 10 times.
   a. Calculate the expected sum of the 10 rolls.
   b. Calculate the expected variance of the 10 rolls.

7. Prove that if $X$ and $Y$ are identically distributed but not necessarily independent, then $\text{Cov}(X + Y, X - Y) = 0$.

8. (page 390, 8.2–8.3) From past experience, a professor knows that the test score of a student taking her final exam is a random variable $X$ with mean $\mu = 75$ and variance $\sigma^2 = 5$.

   a. Let $n$ be the students taking the test, and let $\overline{X} = \frac{1}{n} \sum X_i$ be the average of the scores on their tests. Determine the mean $\mu_{\overline{X}}$ and the variance $\sigma^2_{\overline{X}}$ of $\overline{X}$.

   b. By the Central limit theorem, if $n$ is large, then the mean $\overline{X}$ is approximately normal. Use that to estimate the number of students that would have to take the exam to ensure a probability of at least 0.9 that the class average be within 5 of 75.

9. Short essay question, one or two paragraphs long. Explain what the prior and posterior probabilities are in Bayesian statistics.
<table>
<thead>
<tr>
<th>Distribution</th>
<th>Type</th>
<th>Mass/density function $f(x)$</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Uniform($n$)</strong></td>
<td>D</td>
<td>$1/n$, for $x = 1, 2, \ldots, n$</td>
<td>$(n + 1)/2$</td>
<td>$(n^2 - 1)/12$</td>
</tr>
<tr>
<td><strong>Uniform($a, b$)</strong></td>
<td>C</td>
<td>$\frac{1}{b - a}$, for $x \in [a, b]$</td>
<td>$a + b/2$</td>
<td>$(b - a)^2/12$</td>
</tr>
<tr>
<td><strong>Bernoulli($p$)</strong></td>
<td>D</td>
<td>$f(0) = 1 - p, f(1) = p$</td>
<td>$p$</td>
<td>$p(1 - p)$</td>
</tr>
<tr>
<td><strong>Binomial($n, p$)</strong></td>
<td>D</td>
<td>$\binom{n}{x}p^x(1-p)^{n-x}$, for $x = 0, 1, \ldots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td><strong>Geometric($p$)</strong></td>
<td>D</td>
<td>$q^{x-1}p$, for $x = 1, 2, \ldots$</td>
<td>$1/p$</td>
<td>$(1 - p)/p^2$</td>
</tr>
<tr>
<td><strong>NegativeBinomial($p, r$)</strong></td>
<td>D</td>
<td>$\binom{x-1}{r-1}p^r q^{x-r}$, for $x = r, r + 1, \ldots$</td>
<td>$r/p$</td>
<td>$r(1 - p)/p^2$</td>
</tr>
<tr>
<td><strong>Hypergeometric($N, M, n$)</strong></td>
<td>D</td>
<td>$\binom{M}{x} \binom{N-M}{n-x}$, for $x = 0, 1, \ldots, n$</td>
<td>$np$</td>
<td>$np(1 - p)$</td>
</tr>
<tr>
<td><strong>Poisson($\lambda t$)</strong></td>
<td>D</td>
<td>$\frac{\lambda t}{x!} e^{-\lambda t}$, for $x = 0, 1, \ldots$</td>
<td>$\lambda t$</td>
<td>$\lambda t$</td>
</tr>
<tr>
<td><strong>Exponential($\lambda$)</strong></td>
<td>C</td>
<td>$\lambda e^{-\lambda x}$, for $x \in [0, \infty)$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td><strong>Gamma($\lambda, r$)</strong></td>
<td>C</td>
<td>$\frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}$</td>
<td>$r/\lambda$</td>
<td>$r/\lambda^2$</td>
</tr>
<tr>
<td><strong>Gamma($\alpha, \beta$)</strong></td>
<td>C</td>
<td>$\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$, for $x \in [0, \infty)$</td>
<td>$\alpha \beta$</td>
<td>$a \beta^2$</td>
</tr>
<tr>
<td><strong>Beta($\alpha, \beta$)</strong></td>
<td>C</td>
<td>$\frac{1}{\beta(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$, for $0 \leq x \leq 1$</td>
<td>$\alpha/\alpha + \beta$</td>
<td>$\alpha \beta/(\alpha + \beta)^2(\alpha + \beta + 1)$</td>
</tr>
<tr>
<td><strong>Normal($\mu, \sigma^2$)</strong></td>
<td>C</td>
<td>$\frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$, for $x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td><strong>ChiSquared($\nu$)</strong></td>
<td>C</td>
<td>$\frac{x^{\nu/2-1} e^{x/2}}{2^{\nu/2} \Gamma(\nu/2)}$, for $x \geq 0$</td>
<td>$\nu$</td>
<td>$2\nu$</td>
</tr>
</tbody>
</table>