

Binomial coefficients
 Math 217 Probability and Statistics
 Prof. D. Joyce, Fall 2014

We'll continue our discussion of combinatorics today. We'll look at binomial coefficients which count combinations, the binomial theorem, Pascal's triangle, and multinomial coefficients.

Binomial coefficients and combinations. Combinations are related to partial permutations, but order is disregarded, as you'll see.

Suppose you're considering a set of 4 elements, such as $\{a, b, c, d\}$, and you want to know how many different subsets of 2 elements it has. There are 6, namely $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$. Note that the order the subset is named is irrelevant; the subset $\{a, b\}$ is the same as the subset $\{b, a\}$. For this example, there are 6 combinations of 4 things taken 2 at a time. The notation we'll use for combinations is $\binom{n}{k}$ when choosing n things taken k at a time, read " n choose k ". Thus, $\binom{4}{2} = 6$. You'll see some other notations like $C(n, k)$, ${}_n C_k$, ${}^n C_k$, C_k^n , or C_n^k .

So, how do you compute $\binom{n}{k}$? You can name a combination by a k -permutation, but k -permutations take into consideration the order the items were selected. Each combination of k items will, therefore be named by $k!$ different k -permutations, so we'll have over counted by a factor of $k!$. So divide the $n!/(n - k)!$ different k -permutations by $k!$ to get the actual number of combinations. Thus,

$$\binom{n}{k} = \frac{n!}{k!(n - k)!}$$

Binomial coefficients. The number $\binom{n}{k}$ of combinations of n things chosen k at a time is usually called a *binomial coefficient*. That's because they occur in the expansion of the n^{th} power of a binomial.

A *binomial* is a polynomial with two terms. Let's take the simplest binomial, $x + y$, and write up a table of its powers $(x + y)^n$ for the first few n .

$(x + y)^0 =$	1
$(x + y)^1 =$	$x + y$
$(x + y)^2 =$	$x^2 + 2xy + y^2$
$(x + y)^3 =$	$x^3 + 3x^2y + 3xy^2 + y^3$
$(x + y)^4 =$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$(x + y)^5 =$	$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

The coefficients in these polynomials, the powers of the binomial $x + y$, are the binomial coefficients. That's the *binomial theorem*.

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^n \frac{n!}{k!(n - k)!} x^k y^{n-k}.$$

To see why binomial coefficients count combinations, consider the coefficient 6 of x^2y^2 . When you expand the product $(x + y)(x + y)(x + y)(x + y)$ you'll get a term x^2y^2 if you choose an x from exactly 2 of the 4 factors $x + y$, the y^2 coming from the other two factors. There are $\binom{4}{2} = 6$ ways of choosing 2 of the four factors, and each one contributes one x^2y^2 , so the coefficient of x^2y^2 in the product will be $\binom{4}{2}$.

One important identity of the many important identities that hold for binomial coefficients is this one:

$$\binom{n}{k} = \binom{n}{n-k}$$

You can see why that's true in three different ways. First, they're both equal to $\frac{n!}{k!(n-k)!}$. Second as coefficients in the expansion of $(x + y)^n$, the coefficient of $x^k y^{n-k}$ is equal to the coefficient of $y^k x^{n-k}$. And third, each subset of k elements in a set of size n has a complement that has $n - k$ elements. The last reason is the best because it directly uses the meaning of $\binom{n}{k}$.

Pascal's triangle. Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665) studied these binomial coefficients in the context of probability in the 1600s. Their correspondence resulted in some of the first significant theory of probability and a systematic study of binomial coefficients. Because of Pascal's publication of their results, a particular arrangement of the binomial coefficients in a triangle is called *Pascal's triangle*. If you prefer, you can call it the *arithmetic triangle*. It was known in Europe for a couple of centuries before Pascal, and it was known much longer in Islamic mathematics, in India, and in China.

Here are the top few rows of the triangle.

				1												
				1		1										
				1		2		1								
				1		3		3		1						
				1		4		6		4		1				
				1		5		10		10		5		1		
				1		6		15		20		15		6		1

The numbers along the sides are all 1s, and each entry in the middle is the sum of the two entries above it. These are just the binomial coefficients $\binom{n}{k}$ arranged in a table. By making it a triangle rather than a rectangle, you can see the two relationships $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and $\binom{n}{k} = \binom{n}{n-k}$ more clearly. Here's the top of the triangle again, but the entries are labelled with binomial coefficients.

										$\binom{0}{0}$												
										$\binom{1}{0}$		$\binom{1}{1}$										
										$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$								
										$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$						
										$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$				
										$\binom{5}{0}$		$\binom{5}{1}$		$\binom{5}{2}$		$\binom{5}{3}$		$\binom{5}{4}$		$\binom{5}{5}$		
										$\binom{6}{0}$		$\binom{6}{1}$		$\binom{6}{2}$		$\binom{6}{3}$		$\binom{6}{4}$		$\binom{6}{5}$		$\binom{6}{6}$

Note that in each row, n is fixed. Let's call that the n th row; the top row is then the 0th row.

There are lots of interrelations among these entries in Pascal's triangle, and we may have time to look at a couple of them. Here's one identity of that kind: the numbers in the n th row sum to 2^n

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

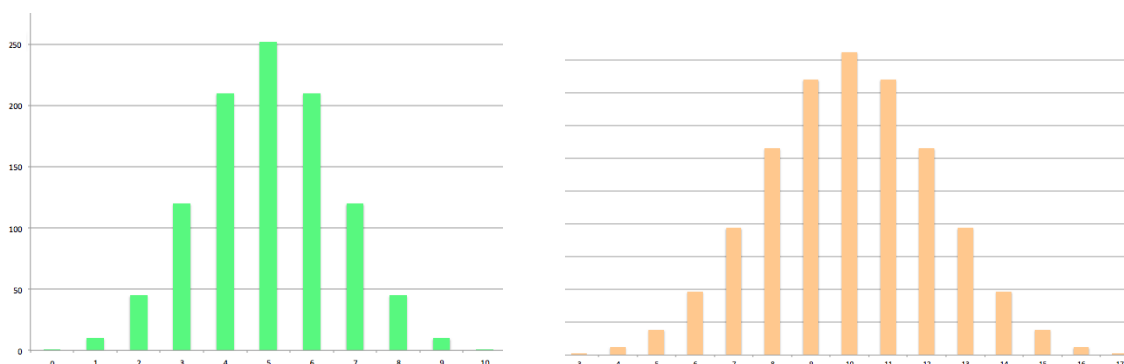
For instance, when $n = 5$, we have $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$.

There are different kinds of proofs you can give for these identities. You can prove this one using counting arguments or algebraic arguments.

Counting proof: These binomial coefficients tell us the number of subsets of various sizes of a set of n elements. Since there are 2^n subsets in all, they have to add up to 2^n .

Algebraic proof: Use the binomial theorem $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ with both x and y set to 1.

It's interesting to look at a row of binomial coefficients displayed as bar chart, or histogram. Compare these graphs for $n = 10$ and $n = 20$.



They're on different scales, and only the center portion of the second graph is shown since the bars are too short to see outside the range shown. Their shapes are about the same. As the course goes on, we'll see a significant role for that shape in the theory of probability.

Multinomial coefficients. The binomial coefficient $\binom{n}{k}$ tells you how many ways you can partition a set of n elements into two subsets, one with k elements and the complement with $n - k$ elements. Multinomial coefficients tell you how many ways you can partition a set of n elements into any number of subsets (not just two) with given cardinalities.

For example, suppose that you've got a set of 15 elements and you want to know how many ways you can partition it into a first set of 5 elements, a second set of 3 elements, a third set of 5 elements, and a fourth set of 2 elements. We can figure that out with binomial coefficients. First choose 5 of the 15 elements in $\binom{15}{5}$ ways, then choose 3 of the remaining 10 elements in $\binom{10}{3}$, then choose 5 of the remaining 7 elements in $\binom{7}{5}$ ways, and the last 2 elements are left for the fourth set. So the answer is

$$\binom{15}{5} \binom{10}{3} \binom{7}{5} = \frac{15!}{5!10!} \frac{10!}{3!7!} \frac{7!}{5!2!} = \frac{15!}{5!3!5!2!}.$$

Note that the numerator is the factorial of the cardinality of the original set, and the denominator is the product of the factorials of cardinalities of the subsets.

In general the number of ways that you can partition a set of n elements into r subsets of cardinalities n_1, n_2, \dots, n_r that sum to n is the *multinomial coefficient*

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$$

Note that the binomial coefficient $\binom{n}{k}$ is the same as the multinomial coefficient $\binom{n}{k, n-k}$. Corresponding to the binomial theorem there is a *multinomial theorem*

$$(x_1 + x_2 + \dots + x_n)^n = \sum_{n_1+n_2+\dots+n_r=n} \binom{n}{n_1, n_2, \dots, n_r} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

where the sum on the right is taken over all nonnegative n_i that sum to n .

We won't need multinomial coefficients as frequently as binomial coefficients, but they will come up on occasion.

Putting k balls into n urns. Suppose that you have k indistinguishable balls that you can put into n distinguishable urns. How many ways can you do that?

You can imagine the urns are placed in a row. You put k_1 balls in the first urn ($k_1 = 0$ is possible), k_2 in the second urn, and so forth, and when you're done you've put all k balls into some urn, so $k_1 + k_2 + \dots + k_n = k$. So we're asking, how many n -tuples (k_1, k_2, \dots, k_n) are there where each k_i is a non negative integer and $k_1 + k_2 + \dots + k_n = k$?

You can find the number in many ways. Here's a clever one. Display each way with 'stars' and 'bars' in a line as follows. Suppose you're putting $k = 6$ balls in $n = 3$ urns according to the triple $(2, 1, 3)$. Draw the 2 balls in the first urn as **, the 1 ball in the second urn as *, and the 3 balls in the the third urn as ***. That gives 6 *'s in all. Separate the urns by bars to give the diagram **|*|***. In fact, every way you can put 6 balls in 3 urns can be described by a list like that of 6 stars and 2 bars. Since there are $\binom{8}{2}$ ways of choosing 2 of the 8 places to be bars and the rest stars, therefore there are $\binom{8}{2}$ ways of putting 6 balls in 3 urns.

More generally, since there are $\binom{n+k-1}{n-1}$, which equals $\binom{n+k-1}{k}$, ways of choosing $n - 1$ places for bars out of $n + k - 1$ places, and putting stars in the other k positions, therefore there are $\binom{n+k-1}{k}$ ways of putting k balls in n urns.

That proof we just finished is typical of combinatorial proofs. Rather than using algebra or calculus, you interpret one kind of thing as another kind of thing, that is, you find a one-to-one correspondence between two sets. Since the two sets have the same size, that gives you a way of counting one kind of thing by turning it into another kind of thing that you already know how to count.

We'll use this kind of combinatorial proof in class to prove what's called Vandermonde's identity, which was first proved in China by Zhu Shijie in 1303:

$$\binom{n+n}{k} = \sum_{j=0}^k \binom{m}{j} \binom{n}{k-j}$$