Sums and Convolution
Math 217 Probability and Statistics
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Today we'll look at the distribution of sums of independent variables. The typical application is where the independent variables have the same distribution, that it, they're a random sample from a distribution, and we want to know the distribution of the sample sum. We'll do it in terms of an operation called *convolution* that gives the distribution for a sum of two independent variables. Repeated application of convolution gives the distribution for a sum of n independent variables.

Sample sums. Let X be a random variable. A sample consists of n independent random variables $X_1, X_2, \ldots, X_n$, each with the same distribution as X. Let $S = X_1 + X_2 + \ldots + X_n$ be the sum of this sample, and

$$\overline{X} = S/n = (X_1 + X_2 + \ldots + X_n)/n$$

be the sample average.

If the mean of X is $\mu = E(X)$ and the variance of X is $\sigma^2 = \mathrm{Var}(X)$, then we saw a couple of meetings ago that the mean and variance of the sample sum S are

$$\mu_S = E(S) = n\mu,$$
$$\sigma_S^2 = \mathrm{Var}(S) = n\sigma^2,$$

while the mean and variance of the sample average $\overline{X}$

$$\mu_{\overline{X}} = E(\overline{X}) = \mu,$$
$$\sigma_{\overline{X}}^2 = \mathrm{Var}(\overline{X}) = \sigma^2/n.$$

The discrete case. Let X and Y be two independent discrete random variables with probability mass functions f_X and f_Y, so that $f_X(x) = P(X{=}x)$ and $f_Y(y) = P(Y{=}y)$. Let $Z = X + Y$. Then

$$f_Z(z) = \sum_y f_X(z-y) f_Y(y)$$

where the sum is taken over all values y for which $f_Y(y) \neq 0$.

This particular operation of creating a new function f_Z by summing values of two other functions f_X and f_Y occurs throughout mathematics, and it's called *convolution*, in particular, f_Z is called the *convolution* of f_X and f_Y, and it's often denoted $f_Z = f_X * f_Y$. Convolution is a commutative and associative operation.

When S is the sample sum, $S = X_1 + X_1 + \cdots + X_n$, of n independent random variables each with the same probability mass function f, then the probability function for S is

$$f_S = f * f * \cdots * f.$$

We've already computed f_S is the one case when X is a Bernoulli random variable with probability p of success and probability $q = 1 - p$ of failure. In that case S is a binomial random variable with $f_S(k) = \binom{n}{k} p^k q^{n-k}$. Finding the distribution of S when X is anything more complicated than a Bernoulli random variable is difficult, but it can be determined with the help of convolutions.

The continuous case. Now let X and Y be two independent continuous random variables with probability density functions f_X and f_Y, and let $Z = X + Y$. Then it can be shown that Z has the probability density function given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy.$$

Although f_Z is determined from f_X and f_Y using integration rather than summation, it is still called the *convolution* of f_X and f_y and denoted $f_Z = f_X * f_Y$. This convolution using integration is also a commutative and associative operation.

The Poisson process. In a Poisson process the gamma distribution is the sum of exponential distributions. When events occur uniformly at random over time at a rate of λ events per unit time, then the random variable T giving the time to the first event has an exponential distribution,

$$f_T(x) = \lambda e^{-\lambda x}, \text{ for } x \in [0, \infty).$$

The gamma distribution gives the time to the r^{th} event. The probability density function for the gamma distribution is

$$f(x) = \frac{1}{(r-1)!} \lambda^r x^{r-1} e^{-\lambda x}, \text{ for } x \in [0, \infty).$$

Normal distributions. Normal distributions are very special in that the sum of normal distributions is another normal distribution. The sum will be a normal distribution whose mean is the sum of the means of all the component distributions, while the variance of the sum is the sum of the variances of all the component distributions.

The text gives a proof in the special case that two standard normal distributions are added together.

An analogous statement holds for Cauchy distributions, but it's more peculiar in that the sum of two Cauchy distributions each with density $f(x) = \dfrac{1}{\pi(1+x^2)}$ is another Cauchy distribution with *exactly the same* density.

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