Math 217 Probability and Statistics

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Return Quiz 1. On chapter 1.

Select first test date.

Due Today. From section 2.2, exercises 1–4.

Due Monday. From section 2.2, exercises 5–7 and 8abc.

Due Wednesday. From section 3.1, exercises 1–6, 12–13.

Last time. More on the concepts of density functions and distribution functions and more examples, Poisson processes and the exponential distribution.

For next time. Read section 3.1 on permutations.

Today. The normal distribution as a limit (see notes from last time), introduction to combinatorics in chapter 3.

Introduction to combinatorics. We're moving on a study of combinatorics, which means counting things. We've seen how when we're studying uniform discrete probabilities, the probability of an event is the ratio of the number of outcomes in the event to the total number of outcomes, and therefore probabilities are reduced to counting elements in a set. But even when we have nonuniform discrete probabilities, counting elements in sets is required. So in this chapter, we'll look at some of the principles of counting things, and, of course, while we're doing that we'll relate the results to questions of probability.

The multiplicative principle, choices and stages, and tree diagrams. Suppose you're making choices in several stages and the number of choices m_i you can choose from at a stage i doesn't depend on previous choices you've made. Then the total number of outcomes for n stages $1, 2, \ldots, n$ is the product $m_1 m_2 \ldots m_n$.

We'll clarify this multiplicative principle with a couple of examples and illustrate them with tree diagrams. Some of the primary applications of this multiplicative principle are to counting permutations and combinations.

Permutations. There are a couple of different ways you can describe permutations. One fairly abstract definition is that a *permutation* of a set A is a one-to-one mapping σ of A to itself. That means that σ is a function $A \to A$ that has an inverse function σ^{-1} such that

$$\sigma(x) = y \quad \text{if and only if} \quad x = \sigma^{-1}(y).$$

For example, take $A = \{a, b, c\}$ and let σ be the permutation $\sigma(a) = b$, $\sigma(b) = c$, and $\sigma(c) = a$. It would be nice to have a more concise notation for permutations than describing what it does to each element of the set A one at a time. There are a couple of useful notations. One is to write it as a two-line table:

$$\sigma = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}.$$

Another is to specify a standard ordering for the elements of A–such as *abc*–and just list the second line of the the two-line table: *bca*. That describes the permutation as a rearrangement.

Counting permutations. Our main question, an easily answered one, is how may permutations are there on a set A of n elements? If n is small, say 4, then we can list all the permutations. Let's list all the rearrangements of *abcd*.

abcd	*bacd*	*cabd*	*dabc*
abdc	*badc*	*cadb*	*dacb*
acbd	*bcad*	*cbad*	*dbac*
acdb	*bcda*	*cbda*	*dbca*
adbc	*bdac*	*cdab*	*dcab*
adcb	*bdca*	*cdba*	*dcba*

There are 24 of them.

Even when n is not small, it's easy to determine how many permutations there are. We just use the multiplicative principle. In the first stage, choose one of the n elements to go first. In the second stage, there are $n-1$ remaining elements, and choose one of them to go second. At the next stage, chose one of the remaining $n-2$ elements to go next. And so forth until the last stage, when there's only one element left, so it goes last. Thus, the number of permutations of a set of n elements is

$$n(n-1)(n-2)\cdots 2\cdot 1.$$

This last expression is usually abbreviated $n!$ and read "n factorial" or "factorial n" (except by some people who like to say "n shriek" or "n bang").

Thus, there are $4! = 24$ permutations of a set of 4 elements; $3! = 6$ permutations of a set of 3 elements; $2! = 2$ permutations of a set of 2 elements; $1! = 1$ permutations of a set of 1 element; and $0! = 1$ permutations of the empty set $\emptyset$. The last is because the unique function $\emptyset \to \emptyset$ is, by our definition, a permutation.

Sterling's approximation for factorials. The factorial function $n!$ grows very fast with n. It's approximately $n^n e^{-n}\sqrt{2\pi n}$. That's Sterling's approximation. Sterling's approximation is particularly useful when you're examining behavior for large n.

k-permutations. Sometimes we don't want full permutations of a set of n elements, but just partial permutations. If $k \leq n$, a *k-permutation* is an ordered listing of just k elements of a set of n elements. For instance, the 3-permutations of *abcd* are these

abc	*bac*	*cab*	*dab*
abd	*bad*	*cad*	*dac*
acb	*bca*	*cba*	*dba*
acd	*bcd*	*cbd*	*dbc*
adb	*bda*	*cda*	*dca*
adc	*bdc*	*cdb*	*dcb*

while the 2-permutations are these

ab	*ba*	*ca*	*da*
ac	*bc*	*cb*	*db*
ad	*bd*	*cd*	*dc*

We can determine how many k-permutations of a set of n elements there are using the multiplicative principle. In the first stage, choose one of the n elements to go first. In the second stage, there are $n-1$ remaining elements, and choose one of them to go second. At the next stage, chose one of the remaining $n-2$ elements to go next. And so forth until the kth stage, when there are $n-k+1$ remaining elements. Thus, the number of k-permutations of a set of n elements is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

There is no particular standard notation for the number of k-permutations of a set of n elements, but you'll see it denoted $(n)_k$ as in our text, nPk, P^n_k, and various other things. We'll probably just use $n!/(n-k)!$.

The Birthday problem. What's the probability that among 23 randomly chosen people there are at least 2 with the same birthday? It's surprisingly large. To answer this question, we'll make a couple of simplifying assumptions. First, we'll assume there are only 365 possible birthdays. Second, we'll assume each is equally probable. This leads to a situation of uniform discrete probability. An outcome in the sample space consists of an assignment of birthdays to each of the 23 people. Since each of the 23 people can have any of 365 birthdays, there are 365^{23} outcomes in the sample space, each with the same probability.

Rather than find the probability that at least 2 have the same birthday, we'll compute the complementary probability, that is, the probability that all 23 have different birthdays. How many ways can that happen? What we want is a 23-permutation from a set of size 365. There are $365!/342!$ of them. Therefore, the probability that all 23 have different birthdays is $\frac{365!/342!}{365^{23}} = 0.4927028$, which is just under $\frac{1}{2}$. So the probability that at least 2 of them have the same birthday is just over $\frac{1}{2}$.