Math 217 Probability and Statistics

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Wednesday, 5 Dec 2007

Last time. Standardized sums. Examples illustrating the central limit theorem. Confidence intervals. Introduction to moment generating functions.

Today. The ordinary generating function, properties of the moment generating function, some moment generating functions for continuous distributions.

The ordinary generating function. It's

$$h(t) = E(t^X).$$

It determines and is determined by the moment generating function $g(t)$ by the equations

$$g(t) = h(e^t), \quad \text{and} \quad h(t) = g(\log t).$$

We won't be using the ordinary generating function, but I thought I'd bring it to your attention in case you come across it some time in the future.

Properties of moment generating functions.

Translation. If $Y = X + a$, then

$$g_Y(t) = e^{at} g_X(t).$$

Proof: $g_Y(t) = E(e^{Yt}) = E(e^{(X+a)t}) = E(e^{Xt}e^{at}) = e^{at}E(e^{Xt}) = e^{at}g_X(t)$. Q.E.D.

Scaling. If $Y = bX$, then

$$g_Y(t) = g_X(bt).$$

Proof: $g_Y(t) = E(e^{Yt}) = E(e^{(bX)t}) = E(e^{X(bt)}) = g_X(bt)$. Q.E.D.

Standardizing. From the last two properties, if

$$X^* = \frac{X-\mu}{\sigma}$$

is the standardized random variable for X, then

$$G_{X^*}(t) = e^{-\mu t/\sigma} g_X(t/\sigma).$$

Proof: First translate by $-\mu$ to get

$$g_{X-\mu}(t) = e^{-\mu t} g_X(t).$$

Then scale that by a factor of $1/\sigma$ to get

$$\begin{aligned} g_{(X-\mu)/\sigma} &= g_{X-\mu}(t/\sigma) \\ &= e^{-\mu t/\sigma} g_X(t/\sigma) \end{aligned}$$

Q.E.D.

Convolution. If X and Y are independent variables, and $Z = X + Y$, then

$$g_Z(t) = g_X(t)\, g_Y(t).$$

Proof: $g_Z(t) = E(e^{Zt}) = E(e^{(X+Y)t}) = E(e^{Xt}e^{Yt})$. Now, since X and Y are independent, so are e^{Xt} and e^{Yt}. Therefore, $E(e^{Xt}e^{Yt}) = E(e^{Xt})E(e^{Yt}) = g_X(t)g_Y(t)$. Q.E.D.

Note that this property of convolution on moment generating functions implies that for a sample sum $S_n = X_1 + X_2 + \cdots + X_n$, the moment generating function is

$$g_{S_n}(t) = (g_X(t))^n.$$

We can couple that with the standardizing property to determine the moment generating function for the standardized sum

$$S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

Since the mean of S_n is $n\mu$ and its standard deviation is $\sigma\sqrt{n}$, so when it's standardized, we get

$$\begin{aligned} g_{S_n^*}(t) &= e^{-n\mu t/(\sigma\sqrt{n})} g_{S_n}(\tfrac{t}{\sigma\sqrt{n}}) \\ &= e^{-\sqrt{n}\mu t/\sigma} g_{S_n}(\tfrac{t}{\sigma\sqrt{n}}) \\ &= e^{-\sqrt{n}\mu t/\sigma} \left(g_X(\tfrac{t}{\sigma\sqrt{n}})\right)^n \end{aligned}$$

We'll use this result when we prove the central limit theorem

Moments and moment generating functions for continuous distributions. The same definitions for these apply to continuous distributions and discrete ones. That is, if X is any random variable, then its n^{th} moment is

$$\mu_n = E(X^n)$$

and its moment generating function is

$$g_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} \frac{\mu_k}{k!}\, t^k.$$

The only difference is that when you compute them, you use integrals instead of sums, and that's because expectation is defined in terms of integrals rather than sums. Thus,

$$\mu_n = E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

and

$$g(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx.$$

We'll look at three examples of moment generating functions of continuous distributions. First, a uniform distribution on $[0,1]$, second an exponential distribution with parameter λ, and third a standard normal distribution.

The moment generating function for a uniform distribution on $[0,1]$. Let X be uniform on $[0,1]$ so that the probability density function f_X has the value 1 on $[0,1]$ and 0 outside this interval.

Let's first compute the moments.

$$\begin{aligned} \mu_n = E(X^n) &= \int_{-\infty}^{\infty} x^n f_X(x)\, dx \\ &= \int_0^1 x^n\, dx \\ &= \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1} \end{aligned}$$

Next, let's compute the moment generating function.

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx \\ &= \int_0^1 e^{tx}\, dx \\ &= \left.\frac{1}{t} e^{tx}\right|_0^1 = \frac{e^t - 1}{t} \end{aligned}$$

Note that the expression for $g(t)$ does not allow $t = 0$ since there is a t in the denominator. Still $g(0)$ can be evaluated by using L'Hôpital's rule. That rule is actually needed when using the moment generating function $g(t)$ to find the moments, such as $\mu_1 = g'(0)$ and $\mu_2 = g''(0)$.

The moment generating function for an exponential distribution with parameter λ. Recall that when events occur uniformly at random over time at a rate of λ events per unit time, then the random variable X giving the time to the first event has an exponential distribution. The density function for X is $f_X(x) = \lambda e^{-\lambda x}$, for $x \in [0,\infty)$.

Let's compute its moment generating function.

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx \\ &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx \\ &= \lambda \int_0^{\infty} e^{(t-\lambda)x}\, dx \\ &= \left.\lambda \frac{e^{(t-\lambda)x}}{t-\lambda}\right|_0^{\infty} \\ &= \left(\lim_{x\to\infty} \lambda \frac{e^{(t-\lambda)x}}{t-\lambda}\right) - \lambda \frac{e^0}{t-\lambda} \end{aligned}$$

Now if $t < \lambda$, then the limit in the last line is 0, so in that case

$$g(t) = \frac{\lambda}{\lambda - t}.$$

This is a minor, yet important point. The moment generating function doesn't have to be defined for all t. We only need it to be defined for t near 0 because we're only interested in its derivatives evaluated at 0.

The moment generating function for the standard normal distribution. Let Z be a random variable with a standard normal distribution. Its probability density function is

$$f_Z(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

It's moments can be computed from the definition, but it takes repeated applications of integration by parts to compute

$$\mu_n = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x^n e^{-x^2/2}\, dx.$$

We won't do that computation here, but it turns out that when n is odd, the integral is 0, so μ_n is 0 if n is odd. On the other hand when n is even, say $n = 2m$, then it turns out that

$$\mu_{2m} = \frac{(2m)!}{2^m m!}.$$

From these values of all the moments, we can compute the moment generating function.

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} \frac{\mu_n}{n!}\, t^n \\ &= \sum_{m=0}^{\infty} \frac{\mu_{2m}}{(2m)!}\, t^{2m} \\ &= \sum_{m=0}^{\infty} \frac{(2m)!}{2^m m!}\, \frac{1}{(2m)!}\, t^{2m} \\ &= \sum_{m=0}^{\infty} \frac{1}{2^m m!}\, t^{2m} \\ &= e^{t^2/2} \end{aligned}$$

Thus, the moment generating function for the standard normal distribution Z is

$$g_Z(t) = e^{t^2/2}.$$

More generally, if $X = \sigma Z + \mu$ is a normal distribution with mean μ and variance σ^2, then the moment generating function is

$$g_X(t) = \exp(\mu t + \sigma^2 t^2/2).$$