Math 218 Mathematical Statistics

Second Test Answers

March 2006

Scale. 90-100 A, 75-89 B. Median 91.

Problem 1. [20; 10 points each part] Recall that the margin of error E of a confidence interval

$$\left[\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

is half the width of the interval, that is, $E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

a. If the level of significance α is fixed, in order to reduce the margin of error E by a factor of 10, does n have to increase or decrease, and by what factor?

E is inversely proportional to $\sqrt{n}$. To reduce E by a factor of 10, $\sqrt{n}$ will have to be increased by a factor of 10, so n will have to be increased by a factor of 100.

(Note that even a moderate decrease in E requires a large increase in n.)

b. If the level of significance α is decreased from 0.10 to 0.05 while the sample size n is left fixed, is the margin of error increased or decreased, and by what factor?

If α is decreased from 0.10 to 0.05, then $z_{\alpha/2}$ will change from $z_{.05} = 1.645$ to $z_{.025} = 1.960$. That's an increase by a factor of $1.960/1.645 = 1.191$. Since E is proportional to $z_{\alpha/2}$, therefore E will increase by the same factor of 1.191.

(Thus going from a 90% confidence level to a 95% confidence level only widens the confidence interval by about 19%, not very much. By the way, going from 95% confidence level to a 99% confidence level introduces a factor of $2.575/1.960 = 1.313$, a widening of 31% of the confidence interval. These intervals don't have to be widened very much since the most likely place for the mean μ is right in the middle.)

Problem 2. [30; 10 points each part] Exercise 6.28, page 234. In 1993 a typical American family spent 22% of its after-tax income on clothing, entertainment, and other activities, while the remaining 78% was spent on essentials like housing, food, transportation, health care, and insurance/pensions. It is suggested that because of the increase in the cost of essentials since 1993, the mean percentage income spent on the first three items has decreased. To test this proposition a random sample of 50 households is taken and the percentage spent on these three items is determined. Assume that the percentage varies across families according to a normal distribution with unknown mean μ and known $\sigma = 5\%$.

a. Set up hypothesis on μ. Clearly, H_0 should be $\mu = 22\%$. State what you think H_1 should be (upper one-sided, lower one-sided, or two-sided) and in one sentence explain why.

The question being asked is whether the mean μ has gone down. That suggests a lower one-sided hypothesis $H_1 : \mu < \mu_0$. (Thus, rejecting H_0 amounts to concluding that μ has gone down; not rejecting H_0 means that data are not sufficient to conclude that μ had gone down.)

b. If the average percentage for the random sample of 50 households is 20.5%, would you conclude that this is a significant decrease compared to the 1993 level at the 99% confidence level, i.e., using the level of significance $\alpha = 0.01$?

We reject H_0 if the statistic $z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$ is less than $-z_\alpha$, equivalently, if

$$\overline{x} < \mu_0 - z_\alpha\frac{\sigma}{\sqrt{n}}.$$

In this case $\overline{x} = .205$ while

$$\mu_0 + z_\alpha\frac{\sigma}{\sqrt{n}} = .22 - 2.236 \cdot \frac{0.05}{7.071} = 0.2041$$

Thus, we don't reject H_0, so we don't conclude that there's a significant decrease.

c. Given that the average percentage is 20.5%, what is the P-value, also called the observed level of significance?

The P-value for this lower one-sided hypothesis test is $\Phi(z)$. Since

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{.205 - .22}{.05/7.071} = -2.121,$$

therefore the P-value is $\Phi(z) = 0.017$.

(This is consistent with our answer in part b. Since 0.01 is less than the P-value 0.017, we don't reject H_0. However, at only a slightly higher α we would reject H_0.)

Problem 3. [30; 10 points each part] Frequently we have used the statistic

$$Z = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

in the use of inferences about an unknown population mean μ. (Here, $\overline{X}$ is the sample mean, S is the sample standard deviation, and n is the size of the sample.)

a. What are the two approximations that justify the claim that Z is approximately standard normal?

The first approximation is that the sample mean $\overline{X}$ is approximately normal for large n. The second approximation is that for large n, the sample standard deviation S is approximately σ.

b. What theorem is used to justify one at least one of those claims?

The central limit theorem says the sample mean $\overline{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$.

c. What assumption or assumptions are necessary for the justification you gave in part b?

The main assumption that the central limit theorem needs is that n is large. (Of course, to be a sample, the X_i's are independent all with the same distribution. Other assumptions that almost always hold—but not for Cauchy populations—are that the population mean μ and standard deviation σ exist and are finite.)

Problem 4. [20] Exercises 8.7 and 8.16. In a matched pairs design test, to determine whether glaucoma affects the corneal thickness, measurements were made in 8 people affected by glaucoma in one eye but not in the other. The corneal thicknesses in microns were as follows.

Person	1	2	3	4
Eye affected by glaucoma	488	478	480	426
Eye not affected by glaucoma	484	478	492	444
Difference	4	0	−12	−18
Person	5	6	7	8
Eye affected by glaucoma	440	410	458	460
Eye not affected by glaucoma	436	398	464	476
Difference	4	12	−6	−16

From this data the following statistics can be computed. The mean value for the eye affected by glaucoma is $\overline{x} = 455$ while the mean value for the eye unaffected by glaucoma is $\overline{y} = 459$, so the mean difference is $\overline{d} = \overline{x} - \overline{y} = -4$. The sample standard deviation on the difference works out to be $s_d = 10.74$.

Test $H_0 : \mu_1 = \mu_2$ against a two-sided alternative using $\alpha = .10$. What do you conclude? Do you reject H_0 or not?

First note that $n = 8$, a small number, so a t-test is indicated, at least if both the populations are normally distributed, which is a reasonable assumption in this situation. In a paired t-test, we reject H_0 if

$$|t| > t_{n-1,\alpha/2}$$

where $t = \dfrac{\overline{d}}{s_d/\sqrt{n}}$, or, equivalently, if

$$|\overline{d}| > t_{n-1,\alpha/2}\,\frac{s_d}{\sqrt{n}}.$$

Using the first formulation of the condition, $|t| > t_{n-1,\alpha/2}$, we find that

$$t = \frac{\overline{d}}{s_d/\sqrt{n}} = \frac{-4}{10.74/\sqrt{8}} = -1.053$$

while $t_{n-1,\alpha/2} = 1.895$, and since $1.053 < 1.895$, we do not reject H_0, so the information we have is not sufficient to conclude that glaucoma affects corneal thickness.

Alternatively, using the second formulation of the condition, we have

$$\begin{aligned} t_{n-1,\alpha/2}\,\frac{s_d}{\sqrt{n}} &= t_{7,.05}\,\frac{10.74}{\sqrt{8}} \\ &= 1.894 \cdot 3.801 = 7.199 \end{aligned}$$

and since $4 < 7.199$, we do not reject H_0.

Yet another alternative to answering the question is to find the confidence interval. It has endpoints $\overline{d} \pm t_{n-1,\alpha/2}\,\dfrac{s_d}{\sqrt{n}}$, which works out to be $[-11.2, 3.2]$. Since it includes 0, we do not reject H_0.