Math 218, Mathematical Statistics
D Joyce, Spring 2016

Assignment. Chapter 15: 1, 2, 3, 5.

1. Suppose that x is the number of successes in a sample of n Bernoulli trials with success probability p. Show that the maximum likelihood estimator of p is $\hat{p} = x/n$.

The likelihood function is

$$L(p \mid x) = p^x (1-p)^{n-x}.$$

Its log is

$$\ln L(p \mid x) = x \ln p + (n-x) \ln(1-p).$$

The derivative of that log is

$$\frac{d}{dp} \ln L(p \mid x) = \frac{x}{p} - \frac{n-x}{1-p},$$

so to find the critical points, we solve

$$\frac{x}{p} - \frac{n-x}{1-p} = 0,$$

which gives the solution $p = x/n$. Thus, $\hat{p} = x/n$.

2. Suppose x is the number of trials to achieve the first success in a Bernoulli process with success probability p. Show that the maximum likelihood estimator of p is $\hat{p} = 1/x$.

The distribution for x is geometric with probability mass function $f(x) = p(1-p)^{x-1}$, so that's the likelihood function

$$L(p \mid x) = p(1-p)^{x-1}.$$

Its log is

$$\ln L(p \mid x) = \ln p + (x-1) \ln(1-p)$$

and the derivative of that is

$$\frac{d}{dp} \ln L(p \mid x) = \frac{1}{p} - \frac{x-1}{1-p}$$

Setting that to 0, we find

$$\frac{1}{p} = \frac{x-1}{1-p}$$

which gives the solution $p = 1/x$. Thus $\hat{p} = 1/x$.

3. Suppose that $\mathbf{x}$ is a sample from the exponential distribution with probability density function

$$f(x \mid \theta) = \theta e^{-\theta x} \qquad \text{for } x > 0.$$

Find the maximum likelihood estimator for the mean $\mu = 1/\theta$.

Since $\hat{\mu} = 1/\hat{\theta}$, we can find $\hat{\theta}$ first, then take its reciprocal. (Maximum likelihood estimators are nice that way. If $\hat{\theta}$ is the MLE for θ, then the MLE for any function $f(\theta)$ is $f(\hat{\theta})$.) You could also work with μ instead of θ, but the computations are a bit messier.

The likelihood function is

$$\begin{aligned} L(\theta \mid \mathbf{x}) &= f(x_1|\theta) \cdots f(x_n|\theta) \\ &= \theta e^{-\theta x_1} \cdots \theta e^{-\theta x_n} \\ &= \theta^n e^{-\theta(x_1 + \cdots + x_n)} \end{aligned}$$

Its log is

$$\ln L(\theta \mid \mathbf{x}) = n \ln \theta - \theta(x_1 + \cdots + x_n),$$

and the derivative of that is

$$\frac{d}{d\theta} \ln L(\theta \mid \mathbf{x}) = \frac{n}{\theta} - (x_1 + \cdots + x_n).$$

Setting that to 0 we have

$$\frac{n}{\theta} = x_1 + \cdots + x_n$$

giving the solution $\theta = \dfrac{n}{x_1 + \cdots + x_n} = \dfrac{1}{\overline{x}}$. Since the MLE for θ is the reciprocal of the sample average, $\hat{\theta} = \frac{1}{\overline{x}}$, therefore the MLE of μ is $\hat{\mu} = \overline{x}$.

5. Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is a sample of size n from a Poisson distribution with probability mass function

$$f(x \mid \theta) = \frac{e^{-\theta} \theta^x}{x!}$$

for $x = 0, 1, 2, \ldots$. Think of them as the number of absent employees on n days in a company

a. Find the maximum likelihood function for the parameter θ.

$$\begin{aligned}
L(\theta \,|\, \mathbf{x}) &= \prod_{i=1}^{n} f(x_i \,|\, \theta) \\
&= \prod_{i=1}^{n} \frac{e^{-\theta}\theta_i^x}{x_i!} \\
\ln L(\theta \,|\, \mathbf{x}) &= \sum_{i=1}^{n} -\theta + x_i \ln\theta - \ln(x_i!) \\
&= -n\theta + \sum_{i=1}^{n} x_i \ln\theta - \ln(x_i!) \\
\frac{d}{d\theta} \ln L(\theta \,|\, \mathbf{x}) &= -n + \sum_{i=1}^{n} \frac{x_i}{\theta} \\
&= -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i
\end{aligned}$$

Set that to 0 and solve the equation

$$n = \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

to get $\theta = \overline{x}$.

b. Find the maximum likelihood function of the probability that there is at least one absent employee.

The probability of at least one absent employee is $1 - f(0) = 1 - e^{-\theta}$. So the MLE for this probability is $1 - e^{-\hat{\theta}} = 1 - e^{-\overline{x}}$.

Math 218 Home Page at
`http://math.clarku.edu/~djoyce/ma218/`