Assignment.  Chapter 15: 1, 2, 3, 5.

1. Suppose that \( x \) is the number of successes in a sample of \( n \) Bernoulli trials with success probability \( p \). Show that the maximum likelihood estimator of \( p \) is \( \hat{p} = x/n \).

The likelihood function is
\[
L(p \mid x) = p^x (1-p)^{n-x}.
\]

Its log is
\[
\ln L(p \mid x) = x \ln p + (n-x) \ln(1-p).
\]

The derivative of that log is
\[
\frac{d}{dp} \ln L(p \mid x) = \frac{x}{p} - \frac{n-x}{1-p},
\]
so to find the critical points, we solve
\[
\frac{x}{p} - \frac{n-x}{1-p} = 0,
\]
which gives the solution \( p = x/n \). Thus, \( \hat{p} = x/n \).

2. Suppose \( x \) is the number of trials to achieve the first success in a Bernoulli process with success probability \( p \). Show that the maximum likelihood estimator of \( p \) is \( \hat{p} = 1/x \).

The distribution for \( x \) is geometric with probability mass function \( f(x) = p(1-p)^{x-1} \), so that’s the likelihood function
\[
L(p \mid x) = p(1-p)^{x-1}.
\]

Its log is
\[
\ln L(p \mid x) = \ln p + (x-1) \ln(1-p)
\]
and the derivative of that is
\[
\frac{d}{dp} \ln L(p \mid x) = \frac{1}{p} - \frac{x-1}{1-p}
\]
Setting that to 0, we find
\[
\frac{1}{p} = \frac{x-1}{1-p}
\]
which gives the solution \( p = 1/x \). Thus \( \hat{p} = 1/x \).

3. Suppose that \( x \) is a sample from the exponential distribution with probability density function
\[
f(x \mid \theta) = \theta e^{-\theta x} \quad \text{for } x > 0.
\]

Find the maximum likelihood estimator for the mean \( \mu = 1/\theta \).

Since \( \mu = 1/\hat{\theta} \), we can find \( \hat{\theta} \) first, then take its reciprocal. (Maximum likelihood estimators are nice that way. If \( \hat{\theta} \) is the MLE for \( \theta \), then the MLE for any function \( f(\theta) \) is \( f(\hat{\theta}) \).) You could also work with \( \mu \) instead of \( \theta \), but the computations are a bit messier.

The likelihood function is
\[
L(\theta \mid x) = f(x_1 | \theta) \cdots f(x_n | \theta) = \theta e^{-\theta x_1} \cdots \theta e^{-\theta x_n} = \theta^n e^{-\theta (x_1 + \cdots + x_n)}
\]
Its log is
\[
\ln L(\theta \mid x) = n \ln \theta - \theta (x_1 + \cdots + x_n),
\]
and the derivative of that is
\[
\frac{d}{d\theta} \ln L(\theta \mid x) = \frac{n}{\theta} - (x_1 + \cdots + x_n).
\]
Setting that to 0 we have
\[
\frac{n}{\theta} = x_1 + \cdots + x_n
\]
giving the solution \( \theta = \frac{n}{x_1 + \cdots + x_n} = \frac{1}{\bar{x}} \). Since the MLE for \( \theta \) is the reciprocal of the sample average, \( \bar{\theta} = \frac{1}{\bar{x}} \), therefore the MLE of \( \mu \) is \( \hat{\mu} = \bar{x} \).

5. Suppose that \( x = (x_1, \ldots, x_n) \) is a sample of size \( n \) from a Poisson distribution with probability mass function
\[
f(x \mid \theta) = \frac{e^{-\theta} \theta^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots.
\]

Think of them as the number of absent employees on \( n \) days in a company

a. Find the maximum likelihood function for the parameter \( \theta \).
\[ L(\theta \mid x) = \prod_{i=1}^{n} f(x_i \mid \theta) \]
\[ = \prod_{i=1}^{n} e^{-\theta x_i} x_i! \]
\[ \ln L(\theta \mid x) = \sum_{i=1}^{n} -\theta + x_i \ln \theta - \ln(x_i!) \]
\[ = -n\theta + \sum_{i=1}^{n} x_i \ln \theta - \ln(x_i!) \]
\[ \frac{d}{d\theta} \ln L(\theta \mid x) = -n + \sum_{i=1}^{n} \frac{x_i}{\theta} \]
\[ = -n + \frac{1}{\theta} \sum_{i=1}^{n} x_i \]

Set that to 0 and solve the equation

\[ n = \frac{1}{\theta} \sum_{i=1}^{n} x_i \]

to get \( \theta = \overline{x} \).

b. Find the maximum likelihood function of the probability that there is at least one absent employee.

The probability of at least one absent employee is \( 1 - f(0) = 1 - e^{-\theta} \). So the MLE for this probability is \( 1 - e^{-\theta} = 1 - e^{-\overline{x}} \).

Math 218 Home Page at
http://math.clarku.edu/~djoyce/ma218/