Math 218, Mathematical Statistics

D Joyce, Spring 2016

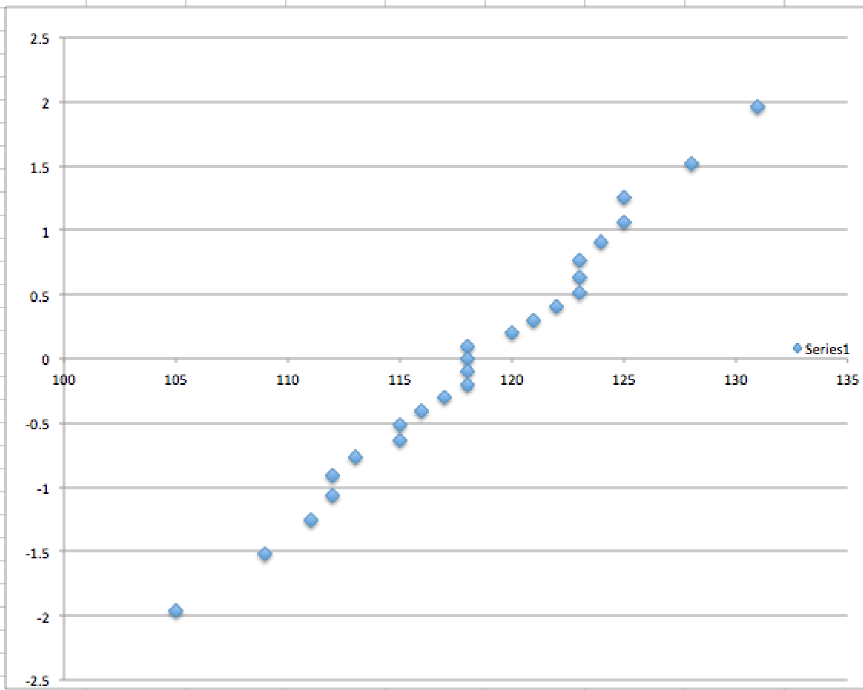
Selected answers. Chapter 7, page 265, exercises 17, 18, 19; and Chapter 8, p. 290, exercises 1–3, 9.

17. Summary of the exercise: There are $n = 25$ tests given with results stated in the problem. Their sample mean is $\overline{x} = 118.5$ and the sample standard deviation is $s = 6.2$.

a. Make a normal probability plot of these readings. Is it reasonable to assume that the data follow a normal distribution?

I used Excel to make my probability plot. There are youtube videos to explain how to do it.

X	Expected Z	i	fi
105	-1.9642168	1	0.02475248
109	-1.5191972	2	0.06435644
111	-1.2593033	3	0.1039604
112	-1.0644416	4	0.14356436
112	-0.9033567	5	0.18316832
113	-0.7628639	6	0.22277228
115	-0.6360367	7	0.26237624
115	-0.5187137	8	0.3019802
116	-0.4081435	9	0.34158416
117	-0.3023618	10	0.38118812
118	-0.1998676	11	0.42079208
118	-0.099436	12	0.46039604
118	0	13	0.5
118	0.09943603	14	0.53960396
120	0.19986756	15	0.57920792
121	0.30236184	16	0.61881188
122	0.40814351	17	0.65841584
123	0.51871372	18	0.6980198
123	0.63603674	19	0.73762376
123	0.76286392	20	0.77722772
124	0.90335666	21	0.81683168
125	1.06444156	22	0.85643564
125	1.25930333	23	0.8960396
128	1.51919718	24	0.93564356
131	1.96421684	25	0.97524752



It looks pretty close to a straight line to me. It doesn't curve up or curve down. It's not S-shaped.

b. Test $H_0 : \sigma \geq 10$ versus $H_1 : \sigma < 10$ at the $\alpha = 0.10$ level. The data for a sample of size $n = 25$ give a sample mean of $\overline{x} = 118.5$ and a sample standard deviation of $s = 6.2$.

This is a lower one-sided hypotheses test on variance. We will reject H_0 if

$$s^2 < \frac{\sigma_0^2 \chi^2_{n-1,1-\alpha}}{n-1}.$$

From the χ^2 table, we see $\chi^2_{24,0.90} = 15.659$, so with the numbers given, that rejection condition says

$$(6.2)^2 < \frac{10^2 \cdot 15.659}{24},$$

which simplifies to $38.44 < 65.25$ which is true, so we reject H_0.

c. Find an upper one-sided 90% confidence interval for σ. Use this interval to test the hypothesis in part **b**.

An upper-one sided confidence interval for σ^2 is

$$\left[0, \frac{(n-1)s^2}{\chi^2_{n-1,\alpha}}\right],$$

so an upper-one sided confidence interval for σ is

$$\left[0, s\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}\right].$$

The right endpoint is $6.2\sqrt{\dfrac{24}{15.659}}$ which equals 7.68. Since $\sigma_0 = 10$ does not lie in this interval, we reject H_0.

18. A bottling company uses a filling machine to fill bottles. A bottle is to contain 475 milliliters (about 16 oz.) of beverage. The actual amount is normally distributed with a standard deviation of 1.0 ml. The purchase of a new machine is contemplated. Based on a sample of 16 bottles filled by the new machine, the sample mean is 476.4 ml and the standard deviation is $s = 0.7$ ml. Does the new machine have a significantly less variation than the current machine?

a. To answer the question posed, test the hypotheses $H_0 : \sigma \geq 1.0$ versus $H_1 : \sigma < 1.0$, where σ is the standard deviation of the new machine. Find bounds on the P-value of the test using table A.5.

We reject this lower one-sided hypothesis H_0 if the sample χ^2 is less than $\chi^2_{n-1,1-\alpha}$. The chi-squared statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{15 \cdot 0.7^2}{1.0^2} = 7.35.$$

Since $\chi^2_{15,0.95} = 7.261$ and $\chi^2_{15,0.90} = 8.547$, therefore the P-value lies between 0.10 and 0.05.

b. Find an appropriate one-sided 95%-confidence interval for σ which can be used to test the hypotheses in part a.

As shown in part a, at the 95%-confidence level we can't reject H_0 (although we could at the 90%-level). But what is the confidence interval? It's the interval

$$\left[s\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}, \infty\right),$$

and since $\chi^2_{15,0.05} = 24.996$, this inteval is

$$\left[0.7\sqrt{\frac{15}{24.996}}, \infty\right) = [0.54, \infty).$$

Note that since this inverval includes 1.0, we can't reject H_0.

19. The durability of upholstery fabric is measured in double rubs (DR), which simulates a person getting in and out of a chair. The manufacturing label on one fabric gives its durability range as 68,000–82,000 DR. The company's quality control department independently evaluated the fabric by testing 25 one-yard samples. The sample mean was 74,283 DR and the sample standard deviation was 4676 DR. Although the mean is acceptable, is the standard deviation consistent with the labelled range? Assume that the DR measurements are normally distributed.

a. Set up the hypotheses to check whether the actual standard deviation is different from the value of σ obtained by dividing the range by four.

As we've seen before, a common way to connect the range of a value to the standard deviation is to take $\frac{1}{4}$ of the range for the standard deviation. Thus, the claim is that the standard deviation is $(82000 - 68000)/4 = 3500$. The null hypothesis is that 3500 is the actual standard deviation, that is, $H_0 : \sigma = 3500$, and the alternate hypothesis is $H_1 : \sigma \neq 3500$. Perhaps, however, a better alternative hypothesis is $H_1 : \sigma > 3500$ because if the claim is a bit conservative, that is, those who made the label listed the range too large to be on the safe side, we should only be looking for the range being too small rather than too large.

b. Test the hypotheses by calculating a 95%-confidence interval for σ. Is the value of σ obtained from the range in the interval? What about a 99%-confidence interval? Summarize your findings.

This answer is for a two-sided confidence interval for σ. If you worked with a one-sided one, then your answer would be different.

The 95%-confidence interval has endpoints

$$\left[s\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}, s\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}}\right].$$

Since $\chi^2_{n-1,\alpha/2} = \chi^2_{24,.025} = 39.364$ and $\chi^2_{n-1,10\alpha/2} = \chi^2_{24,.975} = 12.401$, so this interval is

$$\left[4676\sqrt{\frac{24}{39.364}}, 4676\sqrt{\frac{24}{12.401}}\right] = [3651, 6505].$$

Next, for a 99%-confidence interval. Since $\chi^2_{n-1,\alpha/2} = \chi^2_{24,.005} = 45.558$ and $\chi^2_{n-1,10\alpha/2} = \chi^2_{24,.995} = 9.886$, so this interval is

$$\left[4676\sqrt{\frac{24}{45.558}}, 4676\sqrt{\frac{24}{9.886}}\right] = [3394, 7286].$$

And now the summary. The supposed standard deviation 3500 doesn't lie in the first interval, but does in the second. So, the P-value lies somewere between .05 and .01. You can reject the claim that durability range is 68,000–82,000 DR at the 95%-confidence level, and conclude it's really wider. But you can't at the 99%-confidence level.

Chapter 8

1–2. Are these studies of independent samples or matched pairs? Experimental or observational?

Note that the technical meaning of experimental is that the researcher actively intervenes to control the study conditions.

a. Two computing algorithms are compared in terms of the CPU times required to do the same six problems. Matched pairs since they use the same data sets. Observational.

b. A survey is conducted of teens from inner city schools and surburban schools to compare the proportion who have tried drugs. Independent samples. Observational.

c. A psychologist measures the response times of subjects under two stimuli; each subject is observed under both the stimuli in a random order. Matched pairs. Experimental, the part making it experimental being the random order.

d. An agronomist compares the yields of two varieties of soybean by planting each variety in 10 separate plots of land (a total of 20 plots). Independent samples. Experimental since the assignment to the plots is made by the researcher.

3. Are these studies of independent samples or matched pairs?

a. Military test pilots who had at least one accident are matched by length of experience to pilots without accidents. The two groups are then surveyed about the number of childhood accidents to determine if the former group of pilots is "accident prone." Matched pairs.

b. Lung cancer patients admitted in a hospital over a 12 month period are each matched with a noncancer patient by age, sex, and race. To determine whether or not smoking is a risk factor for lung cancer, it is noted for each patient if he or she is a smoker. Matched pairs.

c. A survey is conducted of college bound and noncollege bound high school seniors to compare the proportion who have at least one parent who attended college. Independent samples.

d. An advertising agency has come up with two different TV commercials for a household detergent. To determine which one is more effective, a test is conducted in which a sample of 100 adults is reandomly divided into two groups. Each group is shown a different commercial, and the people in the group are asked to score the commercial. Independent samples.

9. Two brands of water filters are to be compared in terms of the mean reduction in impurities measured in parts per million (ppm). Twenty-one water samples were tested with each filter and reduction in the impurity level was measured, resulting in the following data:

$$\begin{array}{llll} \text{Filter 1:} & n_1 = 21 & \overline{x}_1 = 8.0 & s_1^2 = 4.5 \\ \text{Filter 2:} & n_2 = 21 & \overline{x}_2 = 6.5 & s_2^2 = 2.0 \end{array}$$

a. Calculate a 95% confidence interval for the mean difference $\mu_1 - \mu_2$ between the two filters, assuming $\sigma_1^2 = \sigma_2^2$. Is there a statistically significant difference at $\alpha = 0.05$ between the two filters?

If each sample is applied to both filters, then this is a matched pairs design, but if there are 42 different samples with 21 applied to each, then it's an independent samples design. It must be that an independent samples design was intended, since s_d is needed for matched pairs and the given information is insufficient to determine s_d.

Here, $\alpha = 0.05$. The two-sided confidence interval has ends

$$(\overline{x} - \overline{y}) \pm t_{n_1+n_2-2,\alpha/2}\, s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$

Here, the pooled sample variance s^2 is the weighted average of the sample variances s_1^2 and s_2^2, but since $n_1 = n_2$, they have the same weight, so $s^2 = \frac{1}{2}(4.5 + 2.0) = 3.25$. Hence, the pooled sample standard deviation is $s = \sqrt{3.25} = 1.803$. From the t-table we have $t_{n_1+n_2-2,\alpha/2} = t_{40,0.025} = 2.021$, so the endpoints are

$$(8.0 - 6.5) \pm 2.021 \cdot 1.803 \sqrt{2/21} = 1.5 \pm 1.12.$$

Thus, the confidence interval is $[0.4, 2.6]$. Since 0 doesn't lie in this interval, there is a statistically significant difference between the two filters at the $\alpha = 0.05$ level.

b. Repeat part **a** without assuming $\sigma_1^2 = \sigma_2^2$. Compare the results.

An approximate two-sided confidence interval for the mean difference $\mu_1 - \mu_2$ has endpoints

$$(\overline{x} - \overline{y}) \pm t_{\nu,\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where

$$\nu = \frac{(w_1 + w_2)^2}{w_1^2/(n_1 - 1) + w_2^2/(n_2 - 1)}$$

with $w_1 = s_1^2/n_1$ and $w_2 = s_2^2/n_2$. So $w_1 = 4.5/21 = 0.214$, $w_2 = 2.0/21 = 0.095$, and

$$\nu = \frac{(0.214 + 0.095)^2}{(0.214)^2/20 + (0.095)^2/20} = 34.8.$$

Since $t_{34.8,0.025}$ is about 2.03, and $\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{2_1 + w^2} = 0.556$, therefore the interval has endpoints 1.5 ± 1.13. We get almost the same confidence interval we found in part **a**.

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