Inferences for two samples
Math 218, Mathematical Statistics
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General questions for statistical inferences. We’ve seen some of the most commonly used point estimators, interval estimates, and confidence intervals. There are many more, and we’ll look at some of them. There are a few general questions to keep in mind when you see a new method for a statistical inference.

• Under what circumstances does the method apply? What kind of sample or samples are under consideration? What are the hypotheses that the method assumes.

• If it’s a hypothesis test, what are the null and alternative hypotheses?

• What is the test statistic? How is it computed from the sample? What kind of distribution does it have?

• How do you apply the method?

Two designs are used when there are two sample populations. Chapter 8 is about comparing means of two populations, one usually being a treatment group, the other a control group.

There are two kinds of these comparisons. One kind is the independent samples design, and for that the individuals in the treatment group are different than the individuals in the control group. In fact, the sizes of the two populations don’t even have to be the same. The other kind is the matched pairs design. Here the individuals are the same in the two populations, but the responses are different, for instance, the control group data may be measured before an experiment, and the treatment data measured after the experiment, but on the same n subjects.

In the first, the independent samples design, we simply have two separate populations and we sample each independently. We assume that the population distributions are of the same kind but have different unknown parameters. The sample sizes, $n_1$ and $n_2$, for the two populations don’t have to be the same but they may be, and it makes sense to have them the same size unless there’s some good reason that one should be smaller.

The second, the matched pairs design, also has two populations, but the samples are not independent. We’ll look at that case next time. The sample sizes for the two populations are the same, and the $i^{th}$ random variable $X_i$ from the first population is not assumed to be independent from $i^{th}$ random variable $Y_i$ from the second population, although they’re assumed to be independent of all the other random variables. For example, $X_i$ might be a pretest score and $Y_i$ the posttest score for the same individual.

Comparing means for two independent samples. There are various tests that apply depending on what is assumed. In all cases we assume that we have two populations where we take a sample

$$X_1, X_2, \ldots, X_{n_1}$$

of size $n_1$ from the first population and one

$$Y_1, Y_2, \ldots, Y_{n_2}$$

of size $n_2$ from the second population. (In practice, $n_1$ is usually equal to $n_2$, but it’s not necessary.) Furthermore, we assume all $n_1 + n_2$ random variables are independent. We’ll denote the population means as $\mu_1$ and $\mu_2$, and the population variances as $\sigma^2_1$ and $\sigma^2_2$. Our job is to make inferences about the population means and/or variances.

Comparing means for large samples. Here we assume that the sample sizes $n_1$ and $n_2$ are large. In this case the sample means $\overline{X}$ and $\overline{Y}$ are
Comparing means for small samples from normal populations. If you assume the two populations are normally distributed, then the difference \(X - \overline{Y}\) is normally distributed even when the sizes of the samples are small. We’ll look at one of tests designed for small samples from normal populations.

In the first test, we assume that the two populations have the same unknown variance \(\sigma^2\). In that case, the test statistic

\[
T = \frac{(X - \overline{Y}) - (\mu_1 - \mu_2)}{S\sqrt{1/n_1 + 1/n_2}}
\]

has a \(t\)-distribution with \((n_1 - 1) + (n_2 - 1) = n - 2\) degrees of freedom, where \(S^2\) is the pooled sample variance given by

\[
S^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)}.
\]

Confidence intervals and hypotheses tests are the same for this test as above, except the test statistic has a \(t\)-distribution instead of a \(z\)-distribution.

There’s a different test when the two populations are not assumed to have the same variance.

Inferences for two samples for matched pair design. Paired \(t\)-tests.

With a matched pairs design, there are two populations, but the samples are not independent. The sample sizes for the two populations are the same \(n\), and the \(i^{th}\) random variable \(X_i\) from the first population is not assumed to be independent from \(i^{th}\) random variable \(Y_i\) from the second population, in fact, we’ll assume that there is a specific correlation \(\rho\) between them. \(\rho = \text{Cor}(X_i, Y_i)\), the same correlation for all \(i\). We do assume, however, that for different subscripts all the random variables are independent.

We’ll also assume that that the populations are normally distributed, \(X_i \sim N(\mu_1, \sigma_1^2)\) and \(Y_i \sim N(\mu_2, \sigma_2^2)\). Then, as in the independent samples design, their differences

\[
D_i = X_i - Y_i
\]
are independent and normally distributed with mean $\mu_D = \mu_1 - \mu_2$ and variance

$$\sigma_D^2 = \text{Var}(D_i) = \text{Var}(X_i - Y_i) = \text{Var}(X_i) + \text{Var}(Y_i) - 2\text{Cov}(X_i, Y_i) = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

Since $D_1, \ldots, D_n$ is sample from a normal distribution $N(\mu_1 - \mu_2, \sigma_D^2)$ where $\sigma_D^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, we can replace $\sigma_D^2$ by the sample variance $S_D^2$ to get a $T$-statistic and then use the $t$-intervals and tests we studied in the last chapter. It follows, for instance, that two-sided $100(1 - \alpha)\%$ confidence level interval for $\mu_D$ has endpoints

$$\bar{d} \pm t_{n-1,\alpha/2} \frac{s_d}{\sqrt{n}}$$

where $\bar{d}$ is the sample average $\frac{1}{n} \sum d_i$ which equal $\bar{x} - \bar{y}$, and $s_d^2$ is the sample variance $\frac{1}{n-1} \sum (d_i - \bar{d})^2$.

As an example hypothesis test, to test the null hypothesis $H_0 : \mu_D = \delta_0$ (where typically $\delta_0$ is 0), against the alternative hypothesis $H_1 : \mu_D \neq \delta_0$, reject $H_0$ when the $t$-statistic

$$t = \frac{\bar{d} - \delta_0}{s_d/\sqrt{n}}$$

has an absolute value greater than $t_{n-1,\alpha/2}$, which is equivalent to the condition

$$|\bar{d} - \delta_0| > t_{n-1,\alpha/2} \frac{s_d}{\sqrt{n}}$$

Hypothesis tests like this are called paired $t$-tests.

The advantage of this matched pairs test over the corresponding independent samples test is that the sample variance of the differences, $s_d^2$, will be much smaller than the pooled sample variance, $s_1^2/n_1 + s_2^2/n_2$. This will make the interval estimates shorter and the hypothesis tests finer.

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