Math 225, Modern Algebra

Midterm

14 Oct 2003

You may refer to the one sheet of notes that you prepared, but nothing else. Write your answers in a bluebook. You may do the problems in any order that you like; just be sure to start each problem on a new page of the bluebook.

Problem 1. On permutations and their matrix representations. [20] Recall that the symmetric group S_3 can be described as the set

$$S_3 = \{1, x, x^2, y, xy, x^2y\},$$

where $x^3 = 1, y^2 = 1$, and $yx = x^2y$.

a. Exhibit a 3×3 matrix that represents the permutation x and another 3×3 matrix that represents y.

b. Using your representation, verify that $yx = x^2y$ in S_3.

c. Show that the permutation y is an odd permutation while x is an even permutation by evaluating their signs.

d. Is the order of y even or odd?

Problem 2. On basic properties of groups. [20] Recall the definition of group that we've used in this course. A *group* is a set G together with a binary operation that satisfies the following three axioms:

(1) Associatvity. $\forall x, y, z$, $(xy)z = x(yz)$.

(2) Identity. $\exists x$, $\forall y$, $xy = y = yx$. Such an x is called an identity element.

(3) Inverses. $\forall x$, $\exists y$, $xy = yx = 1$, where 1 is an identity element as described in axiom 2.

a. Note that axiom 2 says that there is *at least* one identity element. Prove that the identity element is unique. Use only the three axioms for groups mentioned in the defintion, and point out every time you use an axiom in your proof.

b. Prove the cancellation law for groups: if x, y, and z are elements in a group G, then $xz = yz$ implies $x = y$. Use only the three axioms for groups mentioned in the defintion and the results of part a (which allow you to denote the unique group identity as 1) in your proof. Point out every time you use an axiom or part a in your proof.

Problem 3. On the quaternion group. [30] (You don't have to prove your assertions in this problem.)

Recall that one way to describe the quaternion group H is that it contains the eight elements $\pm 1, \pm i, \pm j, \pm k$ and these have the properties

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j.$$

In a homework problem you worked out a multiplication table for H.

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

a. Is H an Abelian group?
b. Is H a cyclic group?
c. Which of the 8 elements is the conjugate iji^{-1}?
d. Name at least one subgroup of H of order 2. Is it normal or not?
e. List all the left cosets of the subgroup that you just named.
f. Name a subgroup of order 4.
g. Is the subgroup you just mentioned a normal subgroup?
h. Are there any quotient groups of H of order 2?
i. Are there any quotient groups of order 4?
j. Is H the product of smaller groups?

Problem 4. On kernels of group homomorphisms. [20]

Let $f : G \to H$ be a group homomorphism, and let $K = \ker f = \{x \in G \mid f(x) = 1\}$ be the kernel of f.

a. Prove that K is a subgroup of G.
b. Prove that K is a normal subgroup of G.

Problem 5. On modular arithmetic. [15] (You don't have to prove your assertions in this problem.)

The set of congruence classes modulo a fixed integer n is usually denoted $\mathbf{Z}/n\mathbf{Z}$. It is written additively, and it's an Abelian group.

a. How many generators of the cyclic group $\mathbf{Z}/10\mathbf{Z}$ are there? Name them.

b. How many subgroups of $\mathbf{Z}/10\mathbf{Z}$ are there? Name them. (You can list their elements if you like.)

c. What are the possible orders of the elements of $\mathbf{Z}/30\mathbf{Z}$? (Of course, 1 and 30 are two orders that occur. What are the others?)