Math 225, Modern Algebra

First Test

Oct 2008

You may use one sheet of notes. Write your answers in the bluebook provided. You may do the problems in any order, but start each problem on a new page.

There are a few extra credit problems. These are meant to be done *after* the test. If you do any, turn them in at our next meeting. (You may use notes and the text for extra credit problems, but don't consult other sources and don't get help from anyone for extra credit problems.)

Problem 1. On ordered fields. [20; 10 points each of two parts] Recall that an *ordered field* consists of a field F along with a subset P whose elements are called *positive* such that

1. F is partitioned into three parts: P, $\{0\}$, and N where
$$N = \{x \in F \mid -x \in P\}$$
the elements of N are called *negative*;
2. the sum of two positive elements is positive; and
3. the product of two positive elements is positive.

Carefully prove the any *two* of the following three properties of ordered fields (you're choice). You may use you know about fields and the definition of ordered field above. For part b you may also use the statement in part a (even if you didn't prove a), and for part c you may use the statements of both part a and part b (even if you didn't prove them).

a. The product $z = xy$ of a negative element x and a positive element y is negative.

b. The product of two negative elements is positive.

c. 1 is positive [hint: $1 \cdot 1 = 1$], and -1 is negative.

(Extra credit: prove all three.)

Problem 2. On rings. [24; 8 points each part] Consider the ring $M_2(\mathbf{R})$ of all 2 by 2 matrices with entries in the real numbers $\mathbf{R}$.

a. Of course the square of both the identity matrix I and the square of its negation $-I$ equal I. Find another 2×2 matrix A whose square is I. (Such matrices are called idempotents.).

(Extra credit: find all the idempotents.)

b. Let S be the set of all these idempotent matrices. Is S a group under addition? (Explain why or why not.)

c. Is S a group under multiplication? (Explain why or why not.)

Problem 3. On groups. [20; 5 points each part] For each of the following, state if it is a group or not. If not, explain why not, but if so, you don't have to give a reason why.

a. The set of 2 by 3 rectangular matrices with integers as entries, where the binary operation is matrix addition.

b. The set of all bijections from the set $S = \{1, 2, 3, 4\}$ to itself. (Recall bijections are also called one-to-one correspondences. A bijection is simultaneously an injection, also called a one-to-one function, and a surjection, also called an onto function.)

c. The set of rational numbers $\mathbf{Q}$ where the binary operation is

$$a * b = a + b + ab.$$

d. The set of real numbers $\mathbf{R}$ where the binary operation is subtraction.

Problem 4. On quaternions. [12] Recall that a quaternion a is an expression

$$x + yi + zj + wk$$

where x, y, z, and w are real numbers and i, j, and k are formal symbols satisfying the properties

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, jk = i, ki = j,$$

and

$$ji = -k, kj = -i, ik = -j.$$

Of course the quaternions $\pm i, \pm j$, and $\pm k$ are six square roots of -1, but there are infinitely many more. Find at least one more. [Hint: look for one where $x = 0$.]

(Extra credit: find all the square roots of -1.)

Problem 5. On noncommutative rings. [24; 8 points each part] We'll make the set $\mathbf{R} \times \mathbf{R}$ into a noncommutative ring by the following definitions of addition and multiplication.

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac, ad + bc) \end{aligned}$$

Note that addition is performed coordinatewise, so it's a group under addition. To show that it's a ring, a few properties have to be verified indluding (1) multiplication is associative, (2) multiplication distributes over addition on the left, and (3) multiplication distributes over addition on the right.

a. Select one of the three and prove it. Your choice.

(Extra credit: prove the others.)

b. What is the multiplicative identity for this ring?

c. Show the ring is not commutative by finding two elements that don't commute.