Math 225 Modern Algebra

First Test Answers

Oct 2008

Scale. 88–98 A. 65–81 B. 41–57 C. Median 84.

Problem 1. On ordered fields. [20; 10 points each of two parts] Recall that an *ordered field* consists of a field F along with a subset P whose elements are called *positive* such that

1. F is partitioned into three parts: P, $\{0\}$, and N where
$$N = \{x \in F \mid -x \in P\}$$
the elements of N are called *negative*;
2. the sum of two positive elements is positive; and
3. the product of two positive elements is positive.

Carefully prove the any *two* of the following three properties of ordered fields (your choice). You may use you know about fields and the definition of ordered field above. For part b you may also use the statement in part a (even if you didn't prove a), and for part c you may use the statements of both part a and part b (even if you didn't prove them).

There are many possible proofs of these statements. I'll give an example proof for each.

a. The product $z = xy$ of a negative element x and a positive element y is negative.

Let x be positive and y be negative. Then $-y$ is positive by condition 1 of the definition. Hence, by condition 3, the product $x(-y)$ is also positive. But that equals $-xy$. Since $-xy$ is positive, therefore xy is negative.

b. The product of two negative elements is positive.

Let x and y be negative. Then $-x$ and $-y$ are positive, and so their product $(-x)(-y)$ is also positive. But that equals xy. Therefore xy is positive.

c. 1 is positive [hint: $1 \cdot 1 = 1$], and -1 is negative.

1 is not 0, so it is either positive or negative. If 1 were negative, then by part b, 1^2 would be positive, but $1^2 = 1$, contradicting 1 being negative. Thus, 1 is positive.

Since 1 is positive, its negation is negative.

(Extra credit: prove all three.)

Problem 2. On rings. [24; 8 points each part] Consider the ring $M_2(\mathbf{R})$ of all 2 by 2 matrices with entries in the real numbers $\mathbf{R}$.

a. Of course the square of both the identity matrix I and the square of its negation $-I$ equal I. Find another 2×2 matrix A whose square is I. (Such matrices are called square roots of 1, or self-inverse matrices.).

(Extra credit: find all the square roots of 1.)

Let the matrix A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A^2 is the matrix
$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}.$$

Thus, in order that $A^2 = I$ the following four equations need to be satisfied.
$$\begin{aligned} a^2 + bc &= 1 \\ b(a + d) &= 0 \\ c(a + d) &= 0 \\ d^2 + bc &= 1 \end{aligned}$$

We can divide the analysis into two cases: case 1 where $a + d \neq 0$, and case 2 where $a + d = 0$.

Case 1. Since $a + d \neq 0$, from the second and third equation we see that $b = c = 0$. Then the first equation becomes $a^2 = 1$ and the fourth equation becomes $d^2 = 1$. Hence, any of the four combinations $a = \pm 1$, $d = \pm 1$ give solutions. In particular, we have two new solutions $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Case 2. In this case $d = -a$, and the second and third equations are satisfied. The first and fourth both become $a^2 + bc = 1$. If b and c are any two real numbers whose product is less than 1, we get a solution $\begin{bmatrix} \pm\sqrt{1 - bc} & b \\ c & \mp\sqrt{1 - bc} \end{bmatrix}$.

b. Let S be the set of all these matrices. Is S a group under addition? (Explain why or why not.)

No, for various reasons. For instance, the additive identity, the zero matrix 0 is not one of them. Also, it's not closed under addition.

c. Is S a group under multiplication? (Explain why or why not.)

It does include the identity matrix I, and it's closed under inverses since every such matrix is its own inverse.

But is it closed under products? Let A and B be such that $A^2 = B^2 = I$. Does $(AB)^2 = I$? That is, does $ABAB = I$? That's equivalent to $AB = BA$ when $A^2 = B^2 = I$. So, do all such matrices commute? No. Take two random matrices from case 2, and they probably won't commute. Also one from case 1 and one from case 2 probably won't commute. (But any two from case 1 do commute.)

Problem 3. On groups. [20; 5 points each part] For each of the following, state if it is a group or not. If not, explain why not, but if so, you don't have to give a reason why.

a. The set of 2 by 3 rectangular matrices with integers as entries, where the binary operation is matrix addition.

Yes.

b. The set of all bijections from the set $S = \{1, 2, 3, 4\}$ to itself. (Recall bijections are also called one-to-one correspondences. A bijection is simultaneously an injection, also called a one-to-one function, and a surjection, also called an onto function.)

I forgot to say what the binary operation was (I meant it to be composition), so I accepted any justified answer.

c. The set of rational numbers $\mathbf{Q}$ where the binary operation is

$$a * b = a + b + ab.$$

Are the rational numbers closed under this operation? Yes.

Is it associative? Yes.

Does it have an identity, that is, is there an element e such that $a * e = e * a = a$ for all a? Yes, namely $e = 0$.

Does it have inverses? That is, if a is any rational number, does there exist an x such that $a * x = 0$? Can we solve $a + x + ax = 0$ for x? Yes, $x = -\dfrac{a}{1+a}$. But wait! If $a = -1$, then such an x doesn't exist.

Therefore, this is not a group. (But it's almost a group.)

d. The set of real numbers $\mathbf{R}$ where the binary operation is subtraction.

No, for various reasons. Subtraction has no identity, that is, there is no number e such that $x - e = e - x = x$ for all x. Also, subtraction is not associative, $(x - y) - z$ does not equal $x - (y - z)$.

Problem 4. On quaternions. [12] Recall that a quaternion a is an expression

$$x + yi + zj + wk$$

where x, y, z, and w are real numbers and i, j, and k are formal symbols satisfying the properties

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, jk = i, ki = j,$$

and

$$ji = -k, kj = -i, ik = -j.$$

Of course the quaternions $\pm i, \pm j$, and $\pm k$ are six square roots of -1, but there are infinitely many more. Find at least one more. [Hint: look for one where $x = 0$.]

Let $a = yi + zj + wk$. Then

$$\begin{aligned}
a^2 &= (yi + zj + wk)(yi + zj + wk) \\
&= y^2 i^2 + yzij + ywik \\
&\quad + zyji + z^2 j^2 + zwjk \\
&\quad + wyki + wzkj + w^2 k^2 \\
&= -y^2 + yzk - ywj \\
&\quad - zyk - z^2 + zwi \\
&\quad + wyj - wzi - w^2 \\
&= -y^2 - z^2 - w^2
\end{aligned}$$

Thus, for a^2 to be -1, we need $-y^2 - z^2 - w^2 = -1$, that is $y^2 + z^2 + w^2 = 1$. Geometrically, that says the point (y, z, w) in 3-space is on the unit sphere.

(Extra credit: find all the square roots of -1.)

The ones just found are all of them, but a proof of that fact would have to consider the cases where $x \neq 0$ too.

Problem 5. On noncommutative rings. [24; 8 points each part] We'll make the set $\mathbf{R} \times \mathbf{R}$ into a ring by the following definitions of addition and multiplication.

$$\begin{aligned}
(a, b) + (c, d) &= (a + c, b + d) \\
(a, b)(c, d) &= (ac, ad + bc)
\end{aligned}$$

Note that addition is performed coordinatewise, so it's a group under addition. To show that it's a ring, a few properties have to be verified indluding (1) multiplication is associative, (2) multiplication distributes over addition on the left, and (3) multiplication distributes over addition on the right.

a. Select one of the three and prove it. Your choice.

Associativity. Check that

$$((a, b)(c, d))(e, f) = (a, b)((c, d)(e, f)).$$

$$\begin{aligned}
((a, b)(c, d))(e, f) &= (ac, ad + bc)(e, f) \\
&= (ace, acf + ade + bce) \\
(a, b)((c, d)(e, f)) &= (a, b)(ce, cf + de) \\
&= (ace, acf + ade + bce)
\end{aligned}$$

Multiplication distributes over addition on the left. Check that $(a, b)((c, d) + (e, f)) = (a, b)(c, d) + (a, b)(e, f)$.

$$\begin{aligned}
(a, b)((c, d) + (e, f)) &= (a, b)(c + e, d + f) \\
&= (ac + ae, ad + af + bc + be) \\
(a, b)(c, d) + (a, b)(e, f) &= (ac, ad + bc) + (ae, af + be) \\
&= (ac + ae, ad + bc + af + be)
\end{aligned}$$

(Extra credit: prove the others.)

b. What is the multiplicative identity for this ring?

We need to find (a, b) so that for all (x, y) we get $(a, b)(x, y) = (x, y)$, that is $(ax, ay + bx) = (x, y)$. Thus, we need both $ax = x$, and $ay + bx = y$. From the first requirement, $a = 1$. The makes the second requirement $y + bx = y$, which simplifies to $bx = 0$. Therefore, $b = 0$. And $(1, 0)$ works as the multiplicative identity for this ring.

c. Show the ring is not commutative by finding two elements that don't commute.

Whoops. It is commutative. Everyone gets 8 points; 16 if you said it was commutative.