Math 225 Modern Algebra

Second Test

November 2017

You may refer to one sheet of notes on this test. Points for each problem are in square brackets. Write your answers in the bluebook provided. You may do the problems in any order that you like, but please start your answers to each problem on a separate page of the bluebook. Please write or print clearly.

1. [20; 10 points each part] Recall that a commutative ring satisfies the cancellation law if whenever $xy = xz$ and $x \neq 0$, then $y = z$. Recall that a commutative ring has no zero-divisors if whenever $xy = 0$, then either $x = 0$ or $y = 0$.

In parts **a** and **b** you'll prove the following theorem: A commutative ring satisfies the cancellation law if and only if it has no zero-divisors.

a. Prove that if a commutative ring R satisfies the cancellation law, then it has no zero-divisors.

b. Prove that if a commutative ring R has no zero-divisors, then it satisfies the cancellation law.

2. [16] The Chinese Remainder Theorem states that if k and m are relatively prime and $n = km$, then $\mathbf{Z}_n \cong \mathbf{Z}_k \times \mathbf{Z}_n$ where an element $[x]_n$ corresponds to the pair $([x]_k, [x]_n)$. In other words, the pair of simultaneous congruences

$$x \equiv a \pmod{k}$$
$$x \equiv b \pmod{m}$$

has a unique solution for x modulo n.

Solve this pair of simultaneous congruences

$$x \equiv 5 \pmod{20}$$
$$x \equiv 20 \pmod{21}$$

for x modulo $20 \cdot 21 = 441$. Use whatever method you like, but show your work.

3. [16] Recall that we defined a Boolean ring as a ring in which every element is idempotent $x^2 = x$, and we proved that Boolean rings are commutative and that $x + x = 0$ holds in a Boolean ring.

Boolean rings correspond to Boolean algebras where multiplication xy in the Boolean ring corresponds to intersection $x \cap y$ in a Boolean algebra, and $x + y + xy$ in the Boolean ring corresponds to union $x \cup y$ in a Boolean algebra.

Since in a Boolean algebra, union distributes over intersection $x \cup (y \cap z) = (x \cap y) \cup (x \cap z)$, therefore the corresponding equation in Boolean rings must also hold. That corresponding equation is

$$x + yz + xyz = (x + y + xy)(x + z + xz).$$

Using those facts about Boolean rings mentioned in the first paragraph above, prove that that equation holds in all Boolean rings.

4. [16] Determine the kernel of the ring homomorphism $f : \mathbf{Z}_{20} \to \mathbf{Z}_5$ where $f([x]_{20}) = [x]_5$. It is enough to list the elements in the kernel.

5. [16] Let R be the polynomial ring $\mathbf{Q}[x]$, and let I be the principal ideal $(x^2 - 3)$ generated by the polynomial $x^2 - 3$. In other words, the elements of I are polynomials of the form $(x^2 - 3)f(x)$ where $f(x)$ is any polynomial in R.

Describe in your own words the quotient ring $R/I = \mathbf{Q}[x]/(x^2 - 3)$. Is it a field?

6. [20; 4 points each part] True/false. For each sentence write the whole word "true" or the whole word "false". If it's not clear whether it should be considered true or false, you may explain in a sentence if you prefer.

a. The natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$ is a ring.

b. Every integral domain is a subring of a field.

c. The product of two fields is a field.

d. If a ring is a finite integral domain, then it is a field.

e. A morphism $f : A \to B$ in a category is defined to be an *isomorphism* if there exists another morphism $g : B \to A$, called its *inverse*, such that $f \circ g = 1_A$.