Exercises
Math 225 Modern Algebra
Fall 2017

3. On rings. None of the following are rings. In each case, the operations of addition and multiplication are the usual ones.

(a). The set of nonzero integers, $\{x \in \mathbf{Z} \,|\, x \neq 0\}$ is not a ring. Why not?

You can come up with a couple reasons why it's not a ring. For one thing, it doesn't have an additive identity. For another, the operation of addition is not defined for all values of the arguments; in particular, $1 + (-1)$ is not an element of the set.

(b). The set of even integers $\{2x \,|\, x \in \mathbf{Z}\}$ is not a ring. Why not?

The only thing that it's missing is a multiplicative identity, 1.

(c). The set of odd degree polynomials with real coefficients

$$\{f(x) \in \mathbf{R}[x] \,|\, \text{the degree of } f(x) \text{ is odd}\}$$

is not a ring. Why not? (How about the set of even degree polynomials?)

The odd degree polynomials are not closed under multiplication, and therefore it's not a ring. For example, x^3 times x^5 is not in the set.

You could come up with other reasons why the odd degree polynomials don't form a ring. When you add $2x^3 + x^2$ to $-2x^3 + 8$, the sum is the even degree polynomial $x^2 + 8$, so it's not closed under addition. Also, the unit 1 is of degree 0, so there is no multiplicative unit.

The even degree polynomials, however, do form a ring. They are closed under addition and multiplication. Constants have degree 0, and 0 is an even number, so the additive identity 0 and the multiplicative identity 1 are even degree polynomials.

4. On noncommutative rings. Are the following rings? (The operations are the usual matrix operations.) Explain in a sentence or two, but a proof is not necessary.

(a). The set of all matrices with real coefficients (all sizes).

No, you can't add matrices of different sizes, and not all of them can be multiplied.

(b). The set of all 2×2 matrices with real entries of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

This one turns out to be a ring. The identity matrix is in that set, it's closed under addition, and it's closed under multiplication. It's worthwhile to check the multiplication.

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bd' \\ 0 & dd' \end{bmatrix}$$

(c). The set of all 2×2 matrices with real entries of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

This also turns out to be a ring. Again, the identity matrix is in the set, and it's clear that it's closed under addition. It's also closed under multiplication.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} = \begin{bmatrix} aa' - bb' & ab' + ba' \\ -ab' - ba' & aa' - bb' \end{bmatrix}$$

This may remind you of multiplication in the complex numbers. In fact, this ring is isomorphic to $\mathbf{C}$ where $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ corresponds to the complex number $a + ib$.

5. Find two matrices in $GL_2(\mathbf{Z})$ that don't commute thereby proving $GL_2(\mathbf{Z})$ is a nonabelian group.

You can hardly miss if you take two random matrices. There are a few matrices that do commute. For example, a scalar matrix (that's a multiple of the identity matrix) will commute with all matrices.

6. Compute the product of $\cos\theta + i\sin\theta$ times $\cos\varphi + i\sin\varphi$. If $x + iy = (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$, then what is x, the real part of the product, in terms of θ and φ? What is y, the imaginary part?

$$\begin{aligned} & (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\ = \; & (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi) \\ = \; & \cos(\theta + \varphi) + i\sin(\theta + \varphi) \end{aligned}$$

So the real part s is the cosine of $\theta + \varphi$ and the imaginary part y is the sine of $\theta + \varphi$.

What this means for complex multiplication is that the angle of the product is the sum of the angles. That's analogous to the formula for logarithms: the log of the product is the sum of the logs. Somehow, complex arithmetic includes logarithms.

Math 225 Home Page at
`http://aleph0.clarku.edu/~djoyce/ma225/`