CLARK UNIVERSITY

Exercises
Math 225 Modern Algebra
Fall 2017

38. As you know, if $n \in \mathbf{Z}$, then $n\mathbf{Z}$, also written (n), is an ideal of the ring $\mathbf{Z}$. Consider the two ideals $I = 6\mathbf{Z}$ and $J = 10\mathbf{Z}$ of the $\mathbf{Z}$.

(a). Determine their intersection $I \cap J$ as a principal ideal of $\mathbf{Z}$.

An integer n lies in intersection $6\mathbf{Z} \cap 10\mathbf{Z}$ if it lies both in $6\mathbf{Z}$ and in $10\mathbf{Z}$. That means both $6\big|n$ and $10\big|n$. That happens if and only if the $\text{LCM}(6,10)\big|n$. Since $\text{LCM}(6,10) = 30$, that means that $30|n$. Thus $6\mathbf{Z} \cap 10\mathbf{Z} = 30\mathbf{Z}$.

(b). Prove that the union $I \cup J$ is not an ideal of $\mathbf{Z}$.

The union $6\mathbf{Z} \cup 10\mathbf{Z}$ consists of all integers that are multiples of 6 or 10 or both. This union does include 0 and is closed under multiples of its elements, so if it's not an ideal, it will have to be because its not closed under addition. Both 6 and 10 lie in the union, but their sum 16 does not lie in the union. Therefore, it's not an ideal.

39. Consider the theorem:

Theorem 1 (Congruence modulo an ideal)**.** Let I be an ideal of a ring R. A congruence, called *congruence modulo* I, is defined by

$$x \equiv y \pmod{I} \text{ if and only if } x - y \in I.$$

The quotient ring, $R/_\equiv$, is denoted R/I.

Prove these two steps required in the proof of the theorem: if $x - x' \in I$ and $y - y' \in I$, then $(x+y) - (x'+y') \in I$ and $(xy - x'y') \in I$.

Proof. Let $x - x' \in I$ and $y - y' \in I$. Since ideals are closed under addition, therefore $x - x' + y - y' \in I$. But $x - x' + y - y' = (x+y) - (x'+y')$, so $(x+y) - (x'+y') \in I$.

Also, $(xy - x'y') = (xy - x'y) + (x'y - x'y') = (x - x')y + x'(y - y')$. Since $(x - x')y \in IR \subseteq I$, and $x'(y - y') \in RI \subseteq I$, their sum $(xy - x'y') \in I$. Q.E.D.

40. Consider the theorem:

Theorem 2. If $f : R \to S$ is a ring homomorphism then the quotient ring $R/\operatorname{Ker} f$ is isomorphic to the image ring $f(R)$, the isomorphism being given by

$$\begin{array}{rcl} R/\operatorname{Ker} f & \to & f(R) \\ x + \operatorname{Ker} f & \mapsto & f(x) \end{array}$$

Prove the preceding theorem.

(a). First show that the assignment $x + \operatorname{Ker} f$ to $f(x)$ is well defined. That means that if $x + \operatorname{Ker} f = x' + \operatorname{Ker} f$, then $f(x) = f(x')$. Call that assignment ϕ.

Suppose that $x + \operatorname{Ker} f = x' + \operatorname{Ker} f$. Then $x - x' \in \operatorname{Ker} f$, so $f(x - x') = 0$. But $f(x - x') = f(x) - f(x')$, so $f(x) = f(x')$.

(b). Show that assignment is a ring homomorphism. Show (1) $\phi(1 + \operatorname{Ker} f) = 1$, (2) $\phi((x + \operatorname{Ker} f) + (y + \operatorname{Ker} f)) = \phi(x + \operatorname{Ker} f) + \phi(y + \operatorname{Ker} f)$, and (3) $\phi((x + \operatorname{Ker} f)(y + \operatorname{Ker} f)) = \phi(x + \operatorname{Ker} f)\phi(y + \operatorname{Ker} f)$.

Part (1): $f(1) = 1$ since f is a ring homomorphism. Therefore $\phi(1) = f(1) = 1$.

Part (2): $\phi((x + \operatorname{Ker} f) + (y + \operatorname{Ker} f)) = \phi(x + y + \operatorname{Ker} f) = f(x+y) = f(x) + f(y) == \phi(x + \operatorname{Ker} f) + \phi(y + \operatorname{Ker} f)$.

Part (3): $\phi((x + \operatorname{Ker} f)(y + \operatorname{Ker} f)) = \phi(xy + \operatorname{Ker} f) = f(xy) = f(x)f(y) = \phi(x + \operatorname{Ker} f)\phi(y + \operatorname{Ker} f)$. Q.E.D.

Since elements of $R/\operatorname{Ker} f$ can be named by elements of R, the notation $x + \operatorname{Ker} f$ is simplified to x. Also, the function ϕ is also denoted f, because with this shortened notation for elements of $R/\operatorname{Ker} f$, $\phi(x) = f(x)$.

41. Prove that R/I is an integral domain if and only if R/I satisfies both conditions (1) $I \neq R$, and (2) $\forall x, y \in R$, if $xy \in I$, then either $x \in I$ or $y \in I$.

Proof. Recall that R/I is an integral domain if it is a commutative ring in which $0 \neq 1$ that satisfies one of the two equivalent conditions: it has no zero-divisors, or it satisfies the cancellation law.

There are various proofs. In each, the condition that $0 \neq 1$ in R/I corresponds to $I \neq R$, and the condition that cancellation and/or zero-divisors in R/I corresponds to $xy \in I$ implying $x \in I$ or $y \in I$.

The condition that $0 \neq 1$ in R/I actually says $0 + I \neq 1 + I$, and that's equivalent to saying $1 \notin I$.

The condition that R/I has no zero-divisors says that if $(x+I)(y+I) = 0+I$, then either $x + I = 0 + I$ or $y + I = 0 + I$. That says $xy + I = I$ (equivalently $xy \in I$) implies $x \in I$ or $y \in I$, and that's condition (2).

Q.E.D.

Math 225 Home Page at
`http://aleph0.clarku.edu/~djoyce/ma225/`