Math Problem Solving Team

Clark University

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After talking about the Putnam Competition we looked at a few problems of the week (POWs), and worked on a couple of them. The cheesecake was good.

Macalester College Problem of the Week. 1081. Who Stole the Book? From http://mathforum.org/wagon/fall07/p1081.html

Six students visited the library on the day a rare book was stolen. Each student entered once, stayed for some time, and left. For any two of them that were in the library at the same time, at least one of them saw the other. The dean questioned the students and learned the following:

Student	Reported seeing
Alice	Bob, Eve
Bob	Alice, Frank
Charlie	Doris, Frank
Doris	Alice, Frank
Eve	Bob, Charlie
Frank	Charlie, Eve

The dean believes that each student reported all the others that he or she saw, with the exception of the thief who, in an attempt to frame another student, reported that other student as being seen when that other student was not in fact in the library. Assume the dean's belief is correct. Can the dean determine the thief?

It took us a while to understand the question, and even longer to decide what assumptions we're supposed to make. From the statement, it seems that if two students were in the library at the same time then either they each saw each other, or one saw the other but the other did not see the one, but it's not possible that each did not see the other. (This seems like a dubious assumption for the dean to make, but that's what it says.) Furthermore, each student will report to the dean all the other students that were seen in the library, even if that student is the thief.

From a later statement, the thief also reported one student he/she didn't see. So 11 of the claimed 12 sightings are correct, and one is false.

Now, Alice said she saw Bob, and Bob said he saw Alice, and at most one of these statements is false. Therefore, Alice's time interval in the library overlaps with Bob's time in the library. Likewise, since Charlie and Frank reported seeing each other, their time intervals in the library overlap.

Next, Alice & Charlie did not report seeing each other, so their time intervals don't overlap. Likewise, there are other pairs whose time intervals don't overlap, namely Alice & Frank, Bob & Charlie, Bob & Doris, Doris & Eve, and Doris & Frank.

In particular, the time when Alice & Bob were both in is disjoint from the time when Charlie & Frank were in.

We also noted that there is a three-cycle where Alice says she saw Eve, Eve says she saw Bob, and Bob says he saw Alice. If these statements are all true, then there was a time when all three were in the library. From the other three-cycle, Charlie–Doris–Frank, if those three statements are all true, then there was also a time when all three of those were in the library. At most one of the 12 statements is false, so at least one of these triple-overlap intervals occurred.

We noted a few other things, perhaps enough to determine the answer to the problem, but thought it would be more interesting to go on to another problem.

Purdue's Problem of the Week. Problem No. 8 (Fall 2007 Series)

Two particles move in the plane so that their positions at time t are $M_t = (1+t, 1+t)$ and $N_t = (t-1, 1-t)$. Let l_t be the line through M_t and N_t. Describe the set S swept out by l_t $\left(\text{i.e., } S = \bigcup_{t=-\infty}^{\infty} l_t\right)$.

The first particle M_t is moving along the line $y = x$ while the second particle is moving along the line $y = -x$, but two units to the left of the first particle.

The line through the two points has slope $\dfrac{(1+t)-(1-t)}{(1+t)-(t-1)}$ which equals t. So the parameter t is the slope of the line. From a quick sketch, it looks like the lines l_t are all tangents to a curve that might be a parabola that opens upward, so the set S swept out by these lines might be the set of curves below that parabola.

A curve whose tangents form a set of lines is said to be the envelope of the set, so it appears that the envelope of this set of lines might be a certain parabola.

We're not yet certain that the envelope is a parabola, but if it is, then from the symmetries of the problem, we can show that its vertex will be the point $(0,1)$ on the y-axis, and the parabola will be one of the form $y = cx^2 + 1$ for some c. It's easy enough to check whether some c works by seeing if the tangent lines to the parabola are all these lines l_t. But this approach of guess and verify just doesn't seem satisfactory. It would be nice to have a method that gives us the answer without having to guess. We can use calculus to find the envelope.

We'll need the equation of the line l_t. Since the slope between a point (x,y) on this line M_t is the same as the slope between (x,y) and N_t, therefore

$$\frac{(1+t)-y}{(1+t)-x} = \frac{(1-t)-y}{(t-1)-x}.$$

A little algebra simplifes this equation for l_t to

$$y = tx + (1-t^2).$$

To find the envelope, we'll take a nearby line l_{t+h}, find where it intersects l_h, then take the limit as $h \to 0$. The intersection point will approach the point of tangency. In particular, the intersection point will be a point on the envelope. The nearby line l_{t+h} has the equation

$$y = (t+h)x + (1-(t+h)^2).$$

These two lines, l_t and l_{t+h}, intersect when $tx + (1 - t^2) = (t+h)x + (1 - (t+h)^2)$, which simplifes to $0 = hx - 2th - h^2$, then to $0 = x - 2t - h$, so they intersect at the point $(x, y) = (2t + h, tx + (1 - t^2))$.

As $h \to 0$, this point will approach the point $(x, y) = (2t, tx + (1 - t^2))$. The set of all such points is the envelope. That is, the envolope is the curve parameterized by t given by the parametric equation $(x, y) = (2t, tx + (1 - t^2))$. Eliminating the parameter t, we find this is the curve $y = \frac{1}{2}x^2 + (1 - \frac{1}{4}x^2)$, that is, $y = 1 + \frac{1}{4}x^2$.