Archimedes' Cattle Problem

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Brief htistory. L. E. Dickson describes the history of Archimede's Cattle Problem in his *History of the Theory of Numbers*,[1] briefly summarized here. A manuscript found in the Herzog August Library at Wolfenbüttel, Germany, was first described in 1773 by G. E. Lessing with a German translation from the Greek and a mathematical commentary by C. Leiste.[2] The problem was stated in a Greek epigram in 24 verses along with a solution of part of the problem, but not the last part about square and triangular numbers. Leiste explained how the partial solution may have been derived, but he didn't solve the last part either. That was first solved by A. Amthor in 1880.[3]

Part 1. We can solve the first part of the problem with a little algebra and a few hand computations. It is a multilinear Diophantine problem with four variables and three equations.

> If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow.

This first part of the problem describes the proportions of bulls of the four different colors. Let W denote the number of white bulls, B the number of black bulls, D the number of dappled bulls, and Y the number of yellow bulls. Then, algebraically, this part says

$$\begin{aligned} W &= (\tfrac{1}{2}+\tfrac{1}{3})B + Y = \tfrac{5}{6}B + Y \\ B &= (\tfrac{1}{4}+\tfrac{1}{5})D + Y = \tfrac{9}{20}D + Y \\ D &= (\tfrac{1}{6}+\tfrac{1}{7})W + Y = \tfrac{13}{42}W + Y \end{aligned}$$

[1] Leonard Eugene Dickson, *History of the Theory of Numbers*, G. E. Stechert, New York, 1934, volume 2, pages 342–345.

[2] *Zur Geschichte der Lieteratur*, Braunschweig, 2, 1773, No. 13, 421–446. Lessing, *Sämmtliche Schriften*, Leipzig, 22, 1802, 221; 9, 1855, 285–302; 12, 1897, 100-115; *Opera*, XIV, 232.

[3] *Zeitschrift Math. Phys.*, 25, 1880, Hist.-Lit. Abt., 153–171.

These three linear equations are easily solved. We have

$$W = \tfrac{5}{6}B + Y = \tfrac{5}{6}(\tfrac{9}{20}D + Y) + Y = \tfrac{5}{6}\left(\tfrac{9}{20}(\tfrac{13}{42}W + Y) + Y\right) + Y.$$

Therefore

$$(1 - \tfrac{5}{6} \cdot \tfrac{9}{20} \cdot \tfrac{13}{42})W = (\tfrac{5}{6} \cdot \tfrac{9}{20} + \tfrac{5}{6} + 1)Y.$$

Clearing the denominators and doing the arithmetic, we find that $4455W = 11130Y$. So a general solution sets $W = 11130t$ and $Y = 4455t$, from which it follows that $D = 7900t$ and $B = 8010t$.

Note that the four coefficients are all divisible by 5, so we can reparametrize the solution as

$$W = 2226t, \quad Y = 891t, \quad D = 1580t, \quad B = 1602t$$

where t is any positive integer. (Since the new coefficients are relatively prime, t cannot be fractional.)

Part 2. The next part of the problem introduces four more variables and four more linear equations. In principle, it should be no harder to solve than part 1, but the numbers are much larger, so it's much easier to make computational errors.

> These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

Let w denote the number of white cows, b the number of black cows, d the number of dappled cows, and y the number of yellow cows. Then, algebraically, this part says

$$\begin{aligned} w &= (\tfrac{1}{3} + \tfrac{1}{4})(B + b) = \tfrac{7}{12}(B + b) \\ b &= (\tfrac{1}{4} + \tfrac{1}{5})(D + d) = \tfrac{9}{20}(D + d) \\ d &= (\tfrac{1}{5} + \tfrac{1}{6})(Y + y) = \tfrac{11}{30}(Y + y) \\ y &= (\tfrac{1}{6} + \tfrac{1}{7})(W + w) = \tfrac{13}{42}(W + w) \end{aligned}$$

These four linear equations should be easily solved for the four new variables w, b, d, and y. We have

$$\begin{aligned} w = \tfrac{7}{12}(B + b) &= \tfrac{7}{12}(B + \tfrac{9}{20}(D + d)) \\ &= \tfrac{7}{12}(B + \tfrac{9}{20}(D + \tfrac{11}{30}(Y + y))) \\ &= \tfrac{7}{12}(B + \tfrac{9}{20}(D + \tfrac{11}{30}(Y + \tfrac{13}{42}(W + w)))) \end{aligned}$$

Therefore

$$(1-\tfrac{7}{12}\cdot\tfrac{9}{20}\cdot\tfrac{11}{30}\cdot\tfrac{13}{42})w = \tfrac{7}{12}\,B+\tfrac{7}{12}\cdot\tfrac{9}{20}\,D+\tfrac{7}{12}\cdot\tfrac{9}{20}\cdot\tfrac{11}{30}\,Y+\tfrac{7}{12}\cdot\tfrac{9}{20}\cdot\tfrac{11}{30}\cdot\tfrac{13}{42}\,W.$$

Clearing the denominators we get

$$\begin{aligned}(12\cdot 20\cdot 30\cdot 42-7\cdot 9\cdot 11\cdot 13)w &= 7\cdot 20\cdot 30\cdot 42\,B\\ &\quad+7\cdot 9\cdot 30\cdot 42\,D\\ &\quad+7\cdot 9\cdot 11\cdot 42\,Y\\ &\quad+7\cdot 9\cdot 11\cdot 13\,W\end{aligned}$$

Therefore,

$$293391w = 176400\,B + 79380\,D + 29106\,Y + 9009\,W.$$

Combining that with our values $W = 2226t$, $Y = 891t$, $D = 1580t$, and $B = 1602t$, we find $293391w = 454000680t$. Thus, $w = \frac{454000680}{293391}t$.

There are a couple of ways to find y, d, and b. One is to find y in terms of w, then d in terms of y, and then b in terms of w. The other is to find them directly. Using the second method, we find

$$\begin{aligned}y &= \tfrac{13}{42}(W+\tfrac{7}{12}(B+\tfrac{9}{20}(D+\tfrac{11}{30}(Y+y))))\\ d &= \tfrac{11}{30}(Y+\tfrac{13}{42}(W+\tfrac{7}{12}(B+\tfrac{9}{20}(D+d))))\\ b &= \tfrac{9}{20}(D+\tfrac{11}{30}(Y+\tfrac{12}{42}(W+\tfrac{7}{12}(B+b))))\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\tfrac{13}{42}\cdot\tfrac{7}{12}\cdot\tfrac{9}{20}\cdot\tfrac{11}{30})y &= \tfrac{13}{42}\,W+\tfrac{13}{42}\cdot\tfrac{7}{12}\,B+\tfrac{13}{42}\cdot\tfrac{7}{12}\cdot\tfrac{9}{20}\,D+\tfrac{13}{42}\cdot\tfrac{7}{12}\cdot\tfrac{9}{20}\cdot\tfrac{11}{30}\,Y\\ (1-\tfrac{11}{30}\cdot\tfrac{13}{42}\cdot\tfrac{7}{12}\cdot\tfrac{9}{20})d &= \tfrac{11}{30}\,Y+\tfrac{11}{30}\cdot\tfrac{13}{42}\,W+\tfrac{11}{30}\cdot\tfrac{13}{42}\cdot\tfrac{7}{12}\,B+\tfrac{11}{30}\cdot\tfrac{13}{42}\cdot\tfrac{7}{12}\cdot\tfrac{9}{20}\,D\\ (1-\tfrac{9}{20}\cdot\tfrac{11}{30}\cdot\tfrac{13}{42}\cdot\tfrac{7}{12})b &= \tfrac{9}{20}\,D+\tfrac{9}{20}\cdot\tfrac{11}{30}\,Y+\tfrac{9}{20}\cdot\tfrac{11}{30}\cdot\tfrac{13}{42}\,W+\tfrac{9}{20}\cdot\tfrac{11}{30}\cdot\tfrac{13}{42}\cdot\tfrac{7}{12}\,B\end{aligned}$$

As before, clearing the denominators gives us

$$\begin{aligned}(42\cdot 12\cdot 20\cdot 30-13\cdot 7\cdot 9\cdot 11)y &= 13\cdot 12\cdot 20\cdot 30\,W\\ &\quad+13\cdot 7\cdot 20\cdot 30\,B\\ &\quad+13\cdot 7\cdot 9\cdot 30\,D\\ &\quad+13\cdot 7\cdot 9\cdot 11\,Y\\ (30\cdot 42\cdot 12\cdot 20-11\cdot 13\cdot 7\cdot 9)d &= 11\cdot 42\cdot 12\cdot 20\,Y\\ &\quad+11\cdot 13\cdot 12\cdot 20\,W\\ &\quad+11\cdot 13\cdot 7\cdot 20\,B\\ &\quad+11\cdot 13\cdot 7\cdot 9\,D\\ (20\cdot 30\cdot 42\cdot 12-9\cdot 11\cdot 13\cdot 7)b &= 9\cdot 30\cdot 42\cdot 12\,D\\ &\quad+9\cdot 11\cdot 42\cdot 12\,Y\\ &\quad+9\cdot 11\cdot 13\cdot 12\,W\\ &\quad+9\cdot 11\cdot 13\cdot 7\,B\end{aligned}$$

Therefore,

$$\begin{aligned} 293391y &= 93600\,B + 54600\,D + 24570\,Y + 9009\,W = 342670419t \\ 293391d &= 110880\,Y + 34320\,W + 20020\,B + 9009\,D = 221496660t \\ 293391b &= 136080\,W + 49896\,B + 15444\,D + 9009\,Y = 308274498t \end{aligned}$$

In summary, the general solution has $w = \frac{454000680}{293391}\,t$, $y = \frac{342670419}{293391}\,t$, $d = \frac{221496660}{293391}\,t$, and $b = \frac{308274498}{293391}\,t$. We need the four quantities w, y, d, and b all to be positive integers, and for that we need to compute some greatest common divisors. All four numerators and the denominator are divisible by 63, so we can rewrite our general solution more simply as $w = \frac{7206360}{4657}\,t$, $y = \frac{5439213}{4657}\,t$, $d = \frac{3515820}{4657}\,t$, and $b = \frac{4893246}{4657}\,t$. Now, 4657 is prime, so we can't reduce reduce the fractions any more. Thus, t must be divisible by 4657 for the numbers of cows of the four different colors to be integers. Let $t = 4657s$, and we have the solution to part 2, namely

$$\begin{aligned} w &= 7206360\,s & W &= 2226\,t = 10366482\,s \\ y &= 5439213\,s & Y &= 891\,t = 4149387\,s \\ d &= 3515820\,s & D &= 1580\,t = 7358060\,s \\ b &= 4893246\,s & B &= 1602\,t = 7460514\,s \end{aligned}$$

where s is any positive integer.

Part 3. The last part of the problem only mentions the bulls. It describes a quadratic Diophantine problem.

> But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

This part says that $W + B$ is a square number, while $Y + D$ is a triangular number. Since $W + B = 2226\,t + 1602\,t = 3828t = 3828 \cdot 4657\,s$, while $Y + D = 891\,t + 158\,t = 2471\,t = 2471 \cdot 4657\,s$, we're looking for a value of s such that $3828t = 3828 \cdot 4657\,s$ is a square number, that is, of the form n^2, at the same time that $2471 \cdot 4657\,s$ is a triangular number, that is, of the form $k(k+1)/2$.

Now for $3828t = 3828 \cdot 4657\,s$ to be a square number, it's enough that $957 \cdot 4657\,s$ be a square number, since $3828 = 957 \cdot 2^2$. And for that, we need $s = 957 \cdot 4657n^2$ for some positive integer n.

Triangular numbers are harder to deal with. We need $2471 \cdot 4657s = 2471 \cdot 957 \cdot 4657^2n^2 = k(k+1)/2$ for some positive integer k. The whole problem boils down to solving the quadratic Diophantine equation

$$2471 \cdot 957 \cdot 4657^2n^2 = k(k+1)/2$$

for k and n. We can complete the square of the right side of this equation:

$$\frac{k^2+k}{2} = \frac{(k+\frac{1}{2})^2 - \frac{1}{4}}{2} = \frac{1}{8}((2k+1)^2 - 1).$$

Now multiply our quadratic Diophantine equation by 8 to put it in this form

$$8 \cdot 2471 \cdot 957 \cdot 4657^2 n^2 = (2k+1)^2 - 1$$

and introduce a new variable $m = 2k+1$ so that it becomes

$$8 \cdot 2471 \cdot 957 \cdot 4657^2 n^2 = m^2 - 1$$

or

$$m^2 - Cn^2 = 1$$

where $C = 410286423278424 = 8{\cdot}2471{\cdot}957{\cdot}4657^2 = 2{\cdot}2471{\cdot}957{\cdot}(2{\cdot}2471)^2 = 4729494{\cdot}(2942)^2$. That equation $m^2 - Cn^2 = 1$ is a classical example of what is now called Pell's equation.

We'll solve it in two stages. First, we'll solve the Pell equation $m^2 - C'n^2 = 1$ where $C' = 4729494$. Second, among the solutions to that equation, we'll find one where n is a multiple of 2942.

Pell's equation and continued fractions. Pell's equation $m^2 - Cn^2 = 1$ has solutions which are the rational approximations to $\sqrt{C}$. Let's look at two examples $C = 19$ and $C = 13$, before taking $C = 8 \cdot 2471 \cdot 957 \cdot 4657^2 = 410286423278424$ as it is for Archimedes' cattle problem.

First, how do you find the continued fraction expansion for $C = 19$, which is about 4.3588989435407? To start with, $\sqrt{19}$ is a bit more than 4. If we let a be reciprocal of that bit more, then $\sqrt{19} = 4 + 1/a$, where $a = 1/0.3588989435 = 2.7862996478$. Next, $a = 2 + 1/b$ where $b = 1/0.7862996478 = 1.2717797887$. Next, $b = 1 + 1/c$ where $c = 1/0.2717797887 = 3.6794494718$. Next, $c = 3 + 1/d$ where $d = 1/0.6794494718 = 1.4717797887$. Next, $d = 1 + 1/e$ where $e = 1/0.4717797887 = 2.1196329812$. Next, $e = 2 + 1/f$ where $f = 1/0.1196329812 = 8.3588989435$. Note that $f - 8 = 0.3588989435 = a - 2$, so $f = 8 + 1/b$. That gives us a continued fraction expansion for $\sqrt{19}$, namely,

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \cdots}}}}}}.$$

To save space this expression can be written either

$$\sqrt{19} = 4 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{1}{8+\cdots}$$

or as

$$\sqrt{19} = [4, 2, 1, 3, 1, 2, 8, \ldots].$$

At this point, the continued fraction expansion repeats, and that is sometimes indicated by overlining the part that repeats, as

$$\sqrt{19} = [4,2,1,3,1,2,8,2,1,3,1,2,8,2,1,3,1,2,8,\ldots] = [4,\overline{2,1,3,1,2,8}].$$

Although the above analysis used numerical computations, an alternate method is to to use algebra.

This continued fraction expansion gives close approximations for $\sqrt{19}$. Its first few approximates are

$$\begin{aligned}
4 &= 4 \\
4+\frac{1}{2} &= \frac{9}{2} \\
4+\frac{1}{2+}\,\frac{1}{1} &= \frac{7}{2} \\
4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3} &= \frac{11}{3} \\
4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1} &= \frac{18}{5} \\
4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2} &= \frac{119}{33}
\end{aligned}$$

These approximates are easy to compute recursively, as in the table below, by using the sequence $[4,\overline{2,1,3,1,2,8}]$. To create the table, write down the first two rows of 0s and 1s as shown, then at the left write the sequence of numbers from the continued fraction. Then compute each new row of the table from the preceeding two rows so that each entry in that new row is found by adding the entry two rows above it and the factor to the left times the entry directly above it. For example, the two entries in the sixth row are $119 = 11 + 6 \cdot 18$ and $33 = 3 + 6 \cdot 5$.

	m	n	m^2	$19n^2$	m^2-19n^2
	0	1			
	1	0			
4	4	1	16	19	−3
2	9	2	81	76	5
1	13	3	169	171	−2
3	48	11	2304	2299	5
1	61	14	3721	3724	−3
2	170	39	28900	28899	1
8	1421	326	2019241	2019244	−3

The left entry m in a row is the numerator of a rational approximate while the right entry n is the denominator.

The interesting thing about these rational approximates of $\sqrt{19}$ is that the square of the numerator m is alternately a little less and a little greater than 19 times the square of the denominator n. The above table has columns for m^2, $19n^2$, and $m^2 - 19n^2$. Note that when $(m,n) = (170,39)$, we have $170^2 - 19 \cdot 38^2 = 1$, which solves the Pell equation $m^2 - 19n^2 = 1$. Indeed, this is the smallest solution to this Pell equation.

Furthermore, all the solutions to the Pell equation can be found from this smallest one, $(m,n)=(170,39)$. You can interpret an equation $m^2-19n^2=1$ as saying that

$$(m+n\sqrt{19})(m-n\sqrt{19})=1,$$

and from that it follows that $(m+n\sqrt{19})^k(m-n\sqrt{19})^k=1$ for any k. This means, for instance, that $(170+39\sqrt{19})(170-39\sqrt{19})=1$ implies, with $k=2$, that $(170+39\sqrt{19})^2(170-39\sqrt{19})^2=1$. Since

$$(170+39\sqrt{19})^2=(170^2+19\cdot 39^2)+(2\cdot 170\cdot 39)\sqrt{19}=57799+13260\sqrt{19},$$

we have a second solution, namely $(m,n)=(57799,13260)$.

Next, let's look at the case $C=13$, which is about 3.605551275464? The continued fraction expression for $\sqrt{13}$ is

$$\sqrt{13}=[3,1,1,1,1,6,1,1,1,1,6,1,1,1,1,6,\ldots]=[3,\overline{1,1,1,1,6}].$$

Here's the table that computes the rational approximates for $\sqrt{13}$.

	m	n	m^2	$13n^2$	m^2-13n^2
	0	1			
	1	0			
3	3	1	9	13	−4
1	4	1	16	13	3
1	7	2	49	52	−3
1	11	3	121	117	4
1	18	5	324	325	−1
6	119	33	14161	14157	4

Note that $18^2-13\cdot 5^2=-1$. That's not quite what we're looking for. We want $m^2-13n^2=+1$. We could go twice as far down the table to find such a solution (m,n), or we could "square" $(18,5)$ to get that solution directly. Since the equation $m^2-13n^2=-1$ says $(m+n\sqrt{13})(m-n\sqrt{13})=-1$, therefore $(m+n\sqrt{13})^2(m-n\sqrt{13})^2=1$. Since

$$(18+5\sqrt{13})^2=(18^2+13\cdot 5^2)+(2\cdot 18\cdot 5)\sqrt{13}=649+180\sqrt{13},$$

we have a solution to $m^2-13n^2=+1$, namely $(m,n)=(649,180)$. In fact, this is the smallest solution, and all the rest can be found by taking powers of it.

Solution to the cattle problem. Let's go back to Archimedes' problem. We have two stages remaining. First, we need to solve the Pell equation $m^2-C'n^2=1$ where $C'=4729494$. Second, among the solutions to that equation, we need to find one where n is a multiple of 2942.

All we need is the continued fraction expansion for $\sqrt{C'}=\sqrt{4729494}$, which is about 2174.7399844579. Here are the computations to find it.

$$
\begin{aligned}
2174.7399844579 &= 2174 + 1/1.35137973413856 \\
1.3513797341386 &= 1 + 1/2.84592394735536 \\
2.8459239473554 &= 2 + 1/1.18213936740570 \\
1.1821393674057 &= 1 + 1/5.49030126898696 \\
5.4903012689870 &= 5 + 1/2.03956233290230 \\
2.0395623329023 &= 2 + 1/25.2765805450719 \\
25.276580545072 &= 25 + 1/3.61574659004589 \\
3.6157465900460 &= 3 + 1/1.62404472256270 \\
1.6240447225627 &= 1 + 1/1.60244925378649 \\
1.6024492537865 &= 1 + 1/1.65989084344423 \\
1.6598908434442 &= 1 + 1/1.51540214557397 \\
1.5154021455740 &= 1 + 1/1.94023251278156 \\
1.9402325127816 &= 1 + 1/1.06356670972973 \\
1.0635667097297 &= 1 + 1/15.7315048120601 \\
15.731504812060 &= 15 + 1/1.36704500573790
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sqrt{4729494} \;=\; & [2174, \overline{1, 2, 1, 5, 2, 25, 3, 1, 1, 1, 1, 1, 1, 15, 1, 2, 16, 1, 2, 1, 1, 8, 6, 1, 21, 1, 1, 3, 1,} \\
& \overline{1, 1, 2, 2, 6, 1, 1, 5, 1, 17, 1, 1, 47, 3, 1, 1, 6, 1, 1, 3, 47, 1, 1, 17, 1, 5, 1, 1, 6, 2, 2, 1,} \\
& \overline{1, 1, 3, 1, 1, 21, 1, 6, 8, 1, 1, 2, 1, 16, 2, 1, 15, 1, 1, 1, 1, 1, 1, 3, 25, 2, 5, 1, 2, 1, 4348}]
\end{aligned}
$$

Let's compute the first few approximates to $\sqrt{C}$.

	m	n
	0	1
	1	0
2174	2174	1
1	2175	1
2	6524	3
1	8699	4
5	50019	23

The smallest solution to $m^2 - Cn^2 = 1$ is found in the row with $m = 81022113$ and $n = 4$, which gives $81022113^2 - 410286423278424 \cdot 4 = 1$.