Notes on Richard Dedekind's *Was sind und was sollen die Zahlen?*

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December 2005

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Introduction

Richard Dedekind (1831–1916) published in 1888 a paper entitled *Was sind und was sollen die Zahlen?* variously translated as *What are numbers and what should they be?* or, as Beman did, *The Nature of Meaning of Numbers*. Dedekind added a second preface to the second edition in 1893. The second edition was translated into English by Wooster Woodruff Beman in 1901.

Much of Dedekind's terminology and notation as translated by Beman differs from the modern standards, so I offer these notes, using current terminology and notation, to make his work more understandable.

Dedekind's essay has 172 numbered paragraphs of 14 sections, here given as a table of contents for the work. After my notes on these sections, I append Beman's translations of Dedekind's two prefaces.

I. Sets and their elements.

¶¶1–20. This section is a short course in basic set theory. Dedekind describes what he means by sets (what Beman called systems) and their elements. He defines subsets, unions, and intersections, and proves the usual properties about them. It's peculiar that he doesn't accept the empty set as a valid set.

¶1. Dedekind uses lowercase letters to name the things [*Dinge*] (soon to be called elements that belong to sets), such as a, b, and c, and he introduces the notation $a = b$ to indicate that a and b denote the same thing. He notes that $a = b$ implies $b = a$, and he notes that $a = b$ and $b = c$ implies $a = c$.

Some of this first paragraph concerns Dedekind's philosophical foundations for his theory. I would just as soon ignore it, but he uses it in paragraph 66 to prove that infinite sets exist. Here's Dedekind's first few sentences of this paragraph, as translated by Beman.

> In what follows I understand by *thing* every object of our thought. In order to be able easily to speak of things, we designate them by symbols, e.g., by letters, and we venture to speak briefly of the thing a of a simply, when we mean the thing denoted by a and not at all the letter a itself. A thing is completely determined by all that can be affirmed or thought concerning it. A thing a is the same as b (identical with b), and b the same as a, when all that can be thought concerning a can also be thought concernting b, and when all that is true of b can also be thought of a.

The explanation of how symbols denote things is just fine; what I object to is his concept of things being objects of our thought. That's an innocent concept, but in paragraph 66 it's used to justify the astounding theorem that infinite sets exist. See Dedekind's Preface to the first edition, appended at the end of these notes, for further explanation of Dedekind's philosophy.

¶2. Dedekind says that he will consider sets [*Systeme*] (translated as system by Beman, but the usual word in English is now set), denoted with uppercase letters such as S and T, that have elements, the things mentioned in paragraph 1. There's a fair amount of philosophy that's of interest for historical reasons, so I'll quote the first half of the paragraph and a footnote as translated by Beman.

> It very frequently happens that different things, $a, b, c, \ldots$ for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a *system* S; we call the things $a, b, c \ldots$ *elements* of the system S, they are *contained* in S; conversely, S *consists* of these elements. Such a system S (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing [1]; it is completely determined when with respect to every thing it is determined whether it is an element of S or not.

This point is footnoted.

> In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances. I mention this expressly because Kronecker not long ago (*Crelle's Journal*, Vol. 99, pp. 334–336) has endeavored to impose certain limitations upon the free formation of concepts in mathematics which I do not believe to be justified; but there seems to be no call to enter upon this matter with more detail until the distinguished mathematician shall have published his reasons for the necessity or merely the expediency of these limitations.

Note that Dedekind says a set is a thing, which allows for a set of sets. Also, with the understanding that a set is determined by its elements, he declares that two sets are equal, $S = T$, when they have exactly the same elements.

He mentions that the set S that contains only one element a should not be considered to be the same as the thing a itself. We would say the singleton set $\{a\}$ is not the same as a. Even though Dedekind uses the same notation for a as for $\{a\}$—indeed he never uses curly braces for sets—I'll make the distinction in these notes.

Interestingly, Dedekind says does not want to consider a set with no elements: "we intend here for certain reasons wholly to exclude the empty system which contains no element at all, although for other investigations it may be appropriate to imagine such a system."

¶3. *Definition.* A set A is a *subset* of a set S, if every element of A is an element of S, written here $A \subseteq S$.

Dedekind uses the same symbol for subset as he does for element membership, but in these notes I'll use $\subseteq$ for subsets and $\in$ for elements since they're more standard.

¶4. *Theorem.* $A \subseteq A$.

The proof depends only on definition 3. I'll note that as [3].

¶5. *Theorem.* If $A \subseteq B$ and $B \subseteq A$, then $A = B$. [2,3]

¶6. *Definition.* A subset A of S is said to be a *proper* subset of S when it does not equal S, that is, when there is an element of S that is not an element of A. [5]

I'll use the notation $A \subset S$ for proper subsets.

¶7. *Theorem.* If $A \subseteq B$ and $B \subseteq C$, which will be denoted $A \subseteq B \subseteq C$, then $A \subseteq C$. If also either $A \subset B$ or $B \subset C$, then $A \subset C$. [3,6]

¶8. *Definition.* The *union* of sets $A, B, C, \ldots$ is the set that has exactly those elements that are in at least one of the sets $A, B, C, \ldots$, that is, as Dedekind says, with his emphasis, "when it is an element of A, *or* B, *or* C,...."

I'll denote this union as $\bigcup\{A, B, C, \ldots\}$, or as $A \cup B$ when there are two sets. Dedekind uses a different symbol than $\bigcup$, but I prefer $\bigcup$ since it's a standard notation now.

He specifically allows a union of one set A, $\bigcup\{A\}$, and identifies it with A itself.

He mentions that the union $\bigcup\{A, B, C, \ldots\}$ is not to be identified with the set whose elements are $A, B, C, \ldots$. This means he allows sets to be elements of other sets.

Dedekind's set theory is naive in the sense that there are no axioms for it. Zermelo developed axioms for set theory some time after this work of Dedekind's. Some sort of axiom is needed to justify the existence of a union of sets. Other axioms are needed, of course, but I don't think it's necessary to mention them, except in paragraph [66].

¶9. *Theorem.* The sets $A, B, C, \ldots$ are each subsets of $\bigcup\{A, B, C, \ldots\}$. [8,3]

¶10. *Theorem.* If the sets $A, B, C, \ldots$ are each subsets of S, then the union $\bigcup\{A, B, C, \ldots\}$ is also a subset of S. [8,3]

¶11. *Theorem.* If the set P is a subset of one of the sets $A, B, C, \ldots$, then P is also a subset of the union $\bigcup\{A, B, C, \ldots\}$. [9,7]

¶12. *Theorem.* If each of the sets $P, Q, \ldots$ is a subset of one of the sets $A, B, C, \ldots$, then the union $\bigcup\{P, Q, \ldots\}$ is also a subset of the union $\bigcup\{A, B, C, \ldots\}$. [11,10]

¶13. *Theorem.* If the set A is the union of some of the sets $P, Q, \ldots$ (not necessarily all of them), then A is a subset of the union $\bigcup\{P, Q, \ldots\}$. [8,3]

¶14. *Theorem.* If each of the sets $A, B, C, \ldots$ is a union of some the sets $P, Q, \ldots$, then the union $\bigcup\{A, B, C, \ldots\}$ is a subset of the union $\bigcup\{P, Q, \ldots\}$. [13,10]

¶15. *Theorem.* If each of the sets $P, Q, \ldots$ is a subset of one of the sets $A, B, C, \ldots$, and if each of the sets $A, B, C, \ldots$ is a union of some of the sets $P, Q, \ldots$, then the union $\bigcup\{A, B, C, \ldots\}$ equals the union $\bigcup\{P, Q, \ldots\}$. [12,14,5]

¶16. *Theorem.* If $A = P \cup Q$ and $B = Q \cup R$, then $A \cup R = P \cup B$.

Proof. Both $A \cup R$ and $P \cup B$ equal $\bigcup\{P, Q, R\}$, by [15]. Q.E.D.

We recognize this as associativity of the union operation:

$$(P \cup Q) \cup R = P \cup (Q \cup R).$$

¶17. *Definition.* A *common* element of the sets $A, B, C, \ldots$ is an element that belongs to each of them. As Dedekind says, with his emphasis, "that is in A *and* in B *and* in $C \ldots$." A set T is a *common* subset of the sets $A, B, C, \ldots$ if it is a subset of each. The *intersection* of sets $A, B, C, \ldots$ is the set that has exactly the common elements of $A, B, C, \ldots$.

I'll denote intersections as $\bigcap(A, B, C, \ldots)$, or as $A \cap B$ when there are two sets. Dedekind uses a different symbol than $\bigcap$, just as he used a different symbol than $\bigcup$ for unions.

Dedekind specifically allows an intersection of one set A and identifies $\bigcap\{A\}$ with A.

Dedekind uses the word *Gemeinheit* for where I put intersection. Beman translated that as community, but probably commonality would have been a better word.

Since he doesn't allow the empty set, he has to say something about intersections when there is no common element. When there is no common element, he says that the symbol $\bigcap\{A, B, C, \ldots\}$ is meaningless. "We shall however almost always in theorems concerning intersections leave it to the reader to add in thought the condition of their existence and to discover the proper interpretation of these theorems for the case of non-existence." It seems to me it would have been easier to accept the empty set as a valid set.

¶18. *Theorem.* Every common subset of the sets $A, B, C, \ldots$ is a subset of the intersection $\bigcap\{A, B, C, \ldots\}$. [17]

¶19. *Theorem.* Every subset of $\bigcap\{A, B, C, \ldots\}$ is a common subset of the sets $A, B, C, \ldots$. [17,7]

¶20. *Theorem.* If each of the sets $P, Q, \ldots$ is a subset of one of the sets $A, B, C, \ldots$, then the intersection $\bigcap\{P, Q, \ldots\}$ is a subset of the intersection $\bigcap\{A, B, C, \ldots\}$.

Proof. Each element in $\bigcap\{P, Q, \ldots\}$ is a common element of $P, Q, \ldots$, therefore a common element of $A, B, C, \ldots$, hence an element of $\bigcap\{A, B, C, \ldots\}$. Q.E.D.

II. Functions on a set.

¶¶21–25. In this section Dedekind defines a function, or transformation, on a set, describes compsition of functions, and develops basic properties of them.

¶21. *Definition.* A *function* (or *transformation*) ϕ with *domain* a set S is a rule that assigns to each element s of S a value $\phi(s)$, called the *image* (or *transform*) of s. We also say ϕ *maps* s to $\phi(s)$.

Dedekind doesn't use the word domain, but just says a function on a set S. But I find that it helps to have a word to refer to the set on which a function is defined.

He also specifically doesn't say to what set $\phi(s)$ belongs, although later in this paragraph, he collects all these images into another set, the image, $\phi(S)$. In section IV, he allows the images of the function to belong to a larger set, and in that section, I'll call that a codomain.

He notes that if T is a subset of S, then the function ϕ with domain S restricts to a function with domain T, which he also denotes ϕ "for the sake of simplicity."

The set $\phi(T)$ is the set of all images $\phi(t)$ for $t \in T$, called the *image* (or *transform*) of T. Thus, $\phi(S)$ is the set of all images $\phi(s)$ for $s \in S$.

He notes that the identity function defined by $\phi(s) = s$ is the simplest function on a set.

For convenience, Dedekind frequently denotes images of a function s' rather than using functional notation $\phi(s)$, as he does here in the rest of the paragraphs of this section.

¶22. *Theorem.* If $A \subseteq B$, then $A' \subseteq B'$.

Proof. Elements in A' are images of elements in A, therefore images of elements in B. Q.E.D.

¶23. *Theorem.* $\phi(\bigcup\{A, B, C, \ldots\}) = \bigcup\{A', B', C', \ldots\}$.

Proof. To show the two sets are equal, we'll show each side is a subset of the other, which suffices according to [5].

Let $M = \phi(\bigcup\{A, B, C, \ldots\}$. Every element m' in M' is the image of an element m in M, therefore, by [8], an element of one of the sets $A, B, C, \ldots$. Hence, m' is an element of one of the sets $A', B', C', \ldots$, and, again by [8], an element of $\bigcup\{A', B', C', \ldots\}$. Therefore, by [3], $\phi(\bigcup\{A, B, C, \ldots\}) \subseteq \bigcup\{A', B', C', \ldots\}$.

On the other hand, $A, B, C, \ldots$ are each subsets of M, by [9], so $A', B', C', \ldots$ are each subsets of M', by [22]. Therefore, by [10], $\bigcup\{A', B', C', \ldots\} \subseteq \phi(\bigcup\{A, B, C, \ldots\})$. Q.E.D.

¶24. *Theorem.* The image of each common subset of $A, B, C, \ldots$, and therefore of the intersection $\bigcap\{A, B, C, \ldots\}$, is a subset of the intersection $\bigcap\{A', B', C', \ldots\}$. [22,18]

¶25. *Definition and theorem.* If ϕ is a function with domain S, and ψ a function with domain $S' = \phi(S)$, then the *composition*, denoted $\psi \circ \phi$, is the function with domain S defined by

$$(\psi \circ \phi)(s) = \psi(\phi(s)).$$

Dedekind doesn't use the $\circ$ symbol for composition preferring juxtaposition.

Dedekind notes that the order of the composition is important, and the reverse composition $\phi \circ \psi$ may not even be defined.

He notes that composition is associative, that is, $\chi \circ (\psi \circ \phi) = (\chi \circ \psi) \circ \phi$, so we're justified in denoting a triple composition without parentheses as $\chi \circ \psi \circ \phi$.

III. One-to-one functions. Similarity of sets.

¶¶26–35. This section starts with the study of one-to-one functions. It also introduces the concept of similarity of sets, which means the two sets are in one-to-one correspondence.

¶26. *Definition.* A function ϕ with domain S is said to be a *one-to-one* function when the images of distinct elements are distinct, that is, when if a does not equal b, then that $a' = \phi(a)$ does not equal $b' = \phi(b)$. Stated positively, $a' = b'$ implies $a = b$.

For a one-to-one function ϕ, there is an *inverse* function ϕ^{-1} with domain $S' = \phi(S)$ defined by $\phi^{-1}(a') = a$. Clearly, ϕ^{-1} is also a one-to-one function. Also, $\phi^{-1}(S') = S$. Moreover, the composition $\phi^{-1} \circ \phi$ is the identity function on S.

The next three paragraphs concern one-to-one functions.

¶27. *Theorem.* If $A' \subseteq B'$, then $A \subseteq B$.

Proof. Suppose $A' \subseteq B'$. Let $a \in A$. Then $a' \in A'$, so $a' \in B'$. Therefore, $a' = b'$ for some $b \in B$. But $a' = b'$ implies $a = b$, therefore, $a \in B$. Thus, $A \subseteq B$. Q.E.D.

¶28. *Theorem.* If $A' = B'$, then $A = B$. [27,4,5]

¶29. If $G = \bigcap\{A, B, C, \ldots\}$, then $G' = \bigcap\{A', B', C', \ldots\}$.

Proof. Each element in $\bigcap\{A', B', C', \ldots\}$ is in S' since S is the domain of ϕ, so it is of the form g' for some $g \in S$. Since g' is a common element of $A', B', C', \ldots$, by [27] g is a common element of $A, B, C, \ldots$, therefore $g \in G$, and so $g' \in G'$. Thus $\bigcap\}A', B', C', \ldots\} \subseteq G'$.

On the other hand, $G' \subseteq \bigcap\{A', B', C', \ldots\}$ by [24].

Hence, by [5] the two sets are equal. Q.E.D.

¶30. *Theorem.* The identity function is a one-to-one function.

¶31. *Theorem.* If ϕ is a one-to-one function with domain S, and ψ is a one-to-one function with domain $\phi(S)$, then the composition $\psi \circ \phi$ is a one-to-one function with domain S, and its inverse is the reverse composition of the inverses of ϕ and ψ, that is, $(\psi \circ \phi)^{-1} = \phi^{-1} \circ \psi^{-1}$.

Proof. If a and b are distinct elements of S, then ϕ maps them to distinct elements $\phi(a)$ and $\phi(b)$ in $\phi(S)$, and then ψ maps those two elements to the distinct elements $(\psi \circ \phi)(a)$ and $(\psi \circ \phi)(b)$ of $(\psi \circ \phi)(S)$. Thus, the composition is one-to-one.

The function $\phi^{-1} \circ \psi^{-1}$ maps each element $(\psi \circ \phi)(s)$ of $(\psi \circ \phi)(S)$ to the element s of S, so it is the inverse of the composition $\psi \circ \phi$. Q.E.D.

¶32. *Definition.* Two sets R and S are said to be *similar*, written $R \sim S$, if there exists a one-to-one function ϕ with domain S such that $\phi(S) = R$, and therefore $\phi^{-1}(R) = S$.

I'll call such a ϕ a *one-to-one correspondence* from R to S.

Dedekind does not have a specific notation for similarity, but it's nice to have one like $R \sim S$.

He notes that any set S is similar to itself, $S \sim S$, since the identity function is a one-to-one correspondence $S \to S$, by [30].

¶33. *Theorem.* If $R \sim S$, then every set Q similar to R is also similar to S.

Proof. If ϕ and ψ are one-to-one correspondences $\phi : S \to R$ and $\psi : R \to Q$, then by [31], $\psi \circ \phi$ is a one-to-one correspondence $S \to Q$. Q.E.D.

This theorem states that similarity is transitive. Although Dedekind doesn't explicitly state it, similarity is also symmetric. That is, $R \sim S$ implies $S \sim R$, since if $\phi : S \to R$ is a one-to-one correspondence, then so is the the inverse function $\phi^{-1} : S \to R$. Thus, similarity is (1) reflexive, $S \sim S$, as Dedekind explicitly noted in 32, and it is (2) symmetric and (3) transitive. Therefore, similarity is an equivalence relation. He uses this result in the next paragraph.

¶34. *Definition.* We can classify sets by putting in one class those sets that are similar to each other. So if $Q, R, S, \ldots$ are similar to a particular set R, then they will all be in the same class, and we may call R a *representative* of that class. According to [33], that class is not changed by choosing a different representative from the same class.

Thus, Dedekind's classes are the equivalence classes of the similarity relation.

¶35. *Theorem.* If $R \sim S$, then every subset of S is similar to a subset of R, and the proper subsets of S are similar to proper subsets of R.

Proof. Let $\phi : S \to R$ be a one-to-one correspondence, and let T be a subset of S. Then by [22], $T \sim \phi(T)$, and $\phi(T) \subseteq R$. Therefore, a subset T of S is similar to the subset $\phi(T)$ of R.

Moreover, if T is a proper subset of S, then there is an element s of S not in T, so by [27], the element $\phi(x)$ is in R, but not in $\phi(T)$, so $\phi(T)$ is a proper subset of R. Q.E.D.

IV. Functions from a set to itself.

¶¶36–63. Given a function $\phi : S \to S$, Dedekind defines a *chain* to be a subset A of S that ϕ maps to itself, that is, $\phi(A) = A$. Given any subset A of S, he defines the chain A_0 generated by A as the intersection of all the chains containing A. After developing all the necessary properties about chains, he states and proves a principle of mathematical induction for chains.

¶36. *Definition.* If ϕ is a function with domain S and $\phi(S)$ is a subset of a set Z, then we say a *a codomain* of ϕ is Z.

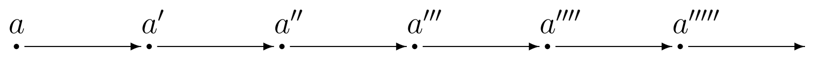
Dedekind doesn't use the words domain or codomain, but says ϕ is a function, or transformation, of S in Z. I'll use the notation $\phi : S \to Z$, and say ϕ maps S to Z.

In this section and many later ones, he's interested in functions $\phi : S \to S$ that map S to itself.

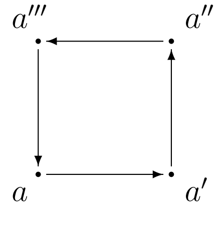
¶37. *Definition.* Given a function $\phi : S \to S$, we say a subset K of S is a *chain* [*Kette*] if ϕ maps K to a subset K' of itself.

Dedekind notes that being a chain is a property of ϕ as much as a property of K.

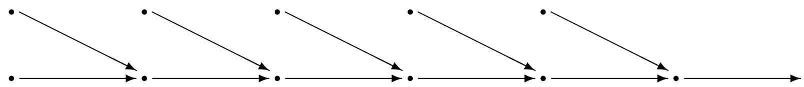
There are no illustrations in *The Nature and Meaning of Numbers*, but a couple of illustrations of chains here would help to explain the concept. The primary example of a chain is a set with distinct elements $a, a', a'', a''', \ldots$.



The sets don't have to be infinite if there are cycles. For instance, if $a'''' = a$, then 4 elements will do.



These two examples both illustrated one-to-one functions, but the function doesn't have to be one-to-one.



Furthermore, although I don't illustrate it, the chains don't even have to be connected. Dedekind shows in paragraph 42 that the union of chains is another chain, and that union may be a disjoint union.

For the following theorems, fix one function ϕ that maps a set S to itself.

¶38. *Theorem.* S itself is a chain.

Although the subject of topology had not yet been invented, and this work on numbers does not suggest in any way that Dedekind was thinking in topological concepts, it's interesting how easily his concepts can be expressed in topological terms.

The underlying set for the topology is the whole set S. This definition of a chain can be interpreted as defining closed sets for the topology. That is, K is closed if $\phi(K) \subseteq K$. Since replacing ϕ by a different function changes the topology, we should include ϕ when denoting the topological space; let's denote it S_ϕ.

With this interpretation, [38] says the entire set S is closed. Theorem [42] says that the union of closed sets is closed, and [43] says that the intersection of closed sets is closed. Definition [44] gives the corresponding closure operation for the topology. It says the chain A_0 of a subset A is the intersection of all the chains containing A; in topological terms, the closure of a subset is the intersection of all closed subsets containing it. Theorem [45] says a subset is contained in its closure. Theorem [47] says if a set A is a subset of a closed set K, then the closure of A is also a subset of that closed set. Theorem [51] says the closure of a closed set is itself. Theorem [54] says that if $A \subseteq B$, then the closure of A is a subset of the closure of B.

Although Dedekind doesn't mention a property corresponding to open sets, he does examine some of these sets beginning in [98]. An subset A is open in S_ϕ if its complement is closed, and that condition can easily shown to be equivalent to the condition $\phi^{-1}(A) \subseteq A$.

¶39. *Theorem.* The image K' of a chain K is itself a chain.
Proof. Since $K' \subseteq K$, therefore by [22], $(K')' \subseteq K'$. Q.E.D.

We can expand this theorem to apply to images under other functions. Suppose we have a commutative diagram of functions.

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S \\ \downarrow^{\psi} & & \downarrow^{\psi} \\ T & \xrightarrow{\theta} & T \end{array}$$

That is, $\psi \circ \phi = \theta \circ \psi$. We can think of ψ as a function $S_\phi \to T_\theta$. Then this theorem can be generalized to say that the image of a chain K in S_ϕ is a chain $\psi(K)$ in T_θ. In topological terms, that says that $\psi : S_\phi \to T_\theta$ is a continuous function.

¶40. *Theorem.* If A is a subset of a chain K, then $A' \subseteq K$.
Proof. $A' \subseteq K' \subseteq K$. [22, 37,7] Q.E.D.

¶41. *Theorem.* If A' is a subset of a chain L, then there is a chain K such that $A \subseteq K$ and $K' \subseteq L$. Indeed, K may be taken to be $A \cup L$.

Proof. Let $K = A \cup L$. Then, by [9], the condition $A \subseteq K$ is satisfied. Also, by [23], $K' = A' \cup L'$. But $A' \subseteq L$, and $L' \subseteq L$, so, by [10], $K' \subseteq L$. Finally, to show K is a chain, since $K' \subseteq L$ and, by [9], $L \subseteq K$, therefore $K' \subseteq K$. Q.E.D.

¶42. *Theorem.* The union of chains is a chain.

Proof. Let M be the union $\bigcup\{A, B, C, \ldots\}$ of chains $A, B, C, \ldots$. Then $A \subseteq A'$, $B \subseteq B'$, $C \subseteq C'$, ... Also, by [23] $M' = \bigcup\{A', B', C', \ldots\}$. Therefore, by [12], $M' \subseteq M$, and M is a chain. Q.E.D.

¶43. *Theorem.* The intersection of chains is a chain.

Proof. Let G be the intersection $\bigcap\{A, B, C, \ldots\}$ of chains $A, B, C, \ldots$. Since, by [17], G is a common subset of $A, B, C, \ldots$, therefore, by [22], G' is a common subset of $A', B', C', \ldots$. Since $A \subseteq A'$, $B \subseteq B'$, $C \subseteq C'$, ..., therefore, by [7] G' is also a common subset of $A, B, C, \ldots$, and, by [18], a subset of G. Thus G is a chain. Q.E.D.

¶44. *Definition.* If A is a subset of S, we will denote by A_0 the intersection of all the chains in S for which A is a subset. This intersection is itself a chain, by [43]. It is called the *chain generated* by the set A, or, more simply, the chain of A.

Dedekind notes that this intersection is nonempty since A is a subset of every chain that contains A. He also notes that S itself is such a chain, so there is at least one such chain.

He emphasizes that A_0, the chain of A, depends on the function ϕ, and when there is more than one function under consideration, then the chain of A for ϕ will be denoted $\phi_0(A)$.

When the set A is a singleton $\{a\}$, then the chain A_0 will be denoted a_0.

¶45. *Theorem.* $A \subseteq A_0$. [18]

¶46. *Theorem.* $(A_0)' \subseteq A_0$. [44,37]

¶47. *Theorem.* If A is a subset of a chain K, then so is A_0.

Proof. A_0 is the intersection of all chains containing A, and among those is K. Q.E.D.

¶48. *Remark.* The smallest chain A_0 containing A is completely characterized by the properties stated in [45,46,47]. Spelled out, A_0 is the subset K of S such that

(1) $A \subseteq K$,
(2) $K' \subseteq K$, and
(3) whenever A is a subset of a chain, then so is K.

¶49. *Theorem.* $A' \subseteq (A_0)'$. [45,22]

¶50. *Theorem.* $A' \subseteq A_0$. [49, 46, 7]

¶51. *Theorem.* If A is a chain, then $A_0 = A$.

Proof. Since A is a subset of the chain A, then by [47], $A_0 \subseteq A$. But [45] says $A \subseteq A_0$, so $A_0 = A$ by [5]. Q.E.D.

¶52. *Theorem.* If $B \subseteq A$, then $B \subseteq A_0$. [45,7]

¶53. *Theorem.* If $B \subseteq A_0$, then $B_0 \subseteq A_0$, and conversely.

Proof. By [47], since A_0 is a chain and $B \subseteq A_0$, therefore $B_0 \subseteq A_0$. Conversely, if $B_0 \subseteq A_0$, then since $B \subseteq B_0$ by [45], therefore $B \subseteq A_0$ by [7]. Q.E.D.

¶54. *Theorem.* If $B \subseteq A$, then $B_0 \subseteq A_0$. [52, 53]

¶55. *Theorem.* If $B \subseteq A_0$, then $B' \subseteq A_0$.

Proof. By [50], $B' \subseteq B_0$, and by [53] $B_0 \subseteq A_0$, so by [7] $B' \subseteq A_0$. Dedekind notes that this can also be proved by [22,46,7] or from [40]. Q.E.D.

¶56. *Theorem.* If $B \subseteq A_0$, then $(B_0)' \subseteq (A_0)'$. [53,22]

¶57. *Theorem and definition.* $(A_0)' = (A')_0$. That is, the image of the chain of A equals the chain of the image of A. We'll denote it A'_0 and call it either the *chain-image* or the *image-chain* of A. An alternative notation for this theorem that explicitly refers to the function ϕ is $\phi(\phi_0(A)) = \phi_0(\phi(A))$.

Proof. As usual, we'll prove two inequalities, which is enough by [5].

Let $L = (A')_0$. Then L is a chain by [44], and by [45] $A' \subseteq L$. Therefore, by [41] there exists a chain K such that $A \subseteq K$ and $K' \subseteq L$. By [47], $A_0 \subseteq K$, so $(A_0)' \subseteq K'$. Hence, by [7] $(A_0)' \subseteq L$, that is $(A_0)' \subseteq (A')_0$.

Now, by [49], $A' \subseteq (A_0)'$, and by [44, 39], $(A_0)'$ is a chain, so by [47], $(A')_0 \subseteq (A_0)'$. Q.E.D.

This theorem can be generalized in the same way that [39] was. Suppose that $\psi : S_\phi \to T_\theta$, that is, $\psi \circ \phi = \theta \circ \psi$, where $\phi : S \to S$, $\theta : T \to T$, and $\psi : S \to T$.

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S \\ \downarrow^{\psi} & & \downarrow^{\psi} \\ T & \xrightarrow{\theta} & T \end{array}$$

Then the image under ψ of the ϕ-chain $\phi_0(A)$ of a subset A of S is the θ-chain $\theta_0(\psi(A))$ of the image $\psi(A)$), that is $\psi(\phi_0(A)) = \theta_0(\psi(A))$.

Here's an argument for $\psi(\phi_0(A)) = \theta_0(\psi(A))$. First the $\subseteq$ half. We need to show $\psi(\phi_0(A)) \subseteq L$ whenever L is a θ-chain containing $\psi(A)$. It suffices to show that there is one ϕ-chain K containing A such that $\psi(A) \subseteq D$. But the chain $C = \phi^{-1}(D)$ works. Next the $\supseteq$ half. $\psi(\phi_0(A))$ is a θ-chain since it's the image of the ϕ-chain $\phi_0(A)$, and it contains $\psi(A)$. But $\theta_0(\psi(A))$ is the smallest θ-chain containing $\psi(A)$. Therefore, $\psi(\phi_0(A)) \supseteq \theta_0(\psi(A))$.

This is stronger than the topological statement. The function $\psi : S_\phi \to T_\theta$ is continuous, so the image a the closure will be contained in the closure of the image, that is, $\psi(\phi_0(A)) \subseteq \theta_0(\psi(A))$, but equality is not required for continuous functions.

¶58. *Theorem.* The chain of a set is the union of that set and its transform chain. That is, $A_0 = A \cup A'_0$.

Proof. By [57] we may let $L = A'_0 = (A_0)' = (A')_0$. Also, let $K = A \cup L$.

As usual, to prove $A_0 = K$, we'll prove two inequalities, which is enough by [5].

By [45], $A' \subseteq L$. Since L is a chain, by [41], so is K. Since $A \subseteq K$ by [9], therefore, by [47], $A_0 \subseteq K$.

On the other hand, by [45] $A \subseteq A_0$, and by [46] $L \subseteq A$), so by [10] $K \subseteq A_0$. Q.E.D.

¶59. *Theorem of mathematical induction.* In order to show that a chain A_0 is a subset of a set Σ (whether or not Σ is a subset of S), it is sufficient to show two things:

ρ. (the base case) that $A \subseteq \Sigma$, and

σ. (the inductive step) that the image of every common element of A_0 and Σ is also an element of Σ.

Proof. Let $G = A_0 \cap \Sigma$. Dedekind notes that ρ implies, by [45], that G is nonempty. Since by [17] $G \subseteq A_0$ and $A_0 \subseteq S$, therefore $G \subseteq S$, so G' is defined. Since $G \subseteq A_0$, therefore, by [55], $G' \subseteq A_0$. Now, G is a common subset of A_0 and Σ, so by σ, $G' \subseteq \Sigma$. Now, since G' is a common subset of A_0 and Σ, therefore, by [18], $G' \subseteq A_0 \cap \Sigma$, that is, $G' \subseteq G$. Thus, G is a chain. But $A \subseteq G$, therefore, by [47], the chain of A is a subset of G, $A_0 \subseteq G$. But also $G \subseteq A_0$, so by [5], $G = A_0$. Therefore, by [17], $A_0 \subseteq \Sigma$. Q.E.D.

¶60. Dedekind says that the preceding theorem is the basis for mathematical induction, the inference from n to $n + 1$. He paraphrases it as follows. In order to prove that all elements of a chain A_0 have a certain property P, it is sufficient to prove

ρ. that all elements of A have the property P, and

σ. that the image n' of every element n of A_0 having property P also has property P.

¶61. *Theorem.* The chain of $\bigcup\{A, B, C, \ldots\}$ is $\bigcup\{A', B', C', \ldots\}$.

Proof. Let $M = \bigcup\{A, B, C, \ldots\}$ and $K = \bigcup\{A', B', C', \ldots\}$.

By [42] K is a chain. By [45] each of the sets $A, B, C, \ldots$ is a subset of one of the sets $A', B', C', \ldots$, so, by [12], $M \subseteq K$.

On the other hand, by [9] each of the systems $A, B, C, \ldots$ is a subset of M, so by [45,7], each is a subset of the chain M_0. Therefore, by [47], each of the chains $A_0, B_0, C_0, \ldots$ is a subset of M_0. Therefore, by [10], $K \subseteq M$.

Thus, by [5], $M = K$. Q.E.D.

¶62. *Theorem.* The chain of $\bigcap\{A, B, C, \ldots\}$ is a subset of $\bigcap\{A', B', C', \ldots\}$.

Proof. Let $G = \bigcap\{A, B, C, \ldots\}$ and $K = \bigcap\{A', B', C', \ldots\}$. By [43], K is a chain. By [45], each of the sets $A, B, C, \ldots$ is a subset of one of the sets $A_0, B_0, C_0, \ldots$, so, by [20], $G \subseteq K$. Therefore, by [47], $G_0 \subseteq K$. Q.E.D.

Note that $\bigcap\{A, B, C, \ldots\}$ may well be empty. If the empty set $\emptyset$ is accepted, then its chain would be itself since $\emptyset' = \emptyset$, so the theorem would apply then, too.

¶63. *Theorem.* If $K' \subseteq L \subseteq K$, then K and L are both chains. Furthermore, if L is a proper subset of K, and U is the subset of all elements of K not in L, and if the chain U_0 is also a proper subset of K, and V is the set of all elements of K not in U_0, then $K = U_0 \cup V$, and $L = U_0' \cup V$. Finally, if $L = K'$, then $V \subseteq V'$.

Dedekind leaves the proof of this theorem to the reader. He notes that he does not use this or the preceding two theorems in the rest of the work.

V. The finite and infinite.

¶¶64–70. This section starts out with Dedekind's famous definition of infinite sets. His proof claiming the existence of an infinite set, however, is curiously nonmathematical, unlike all the rest of this work which is quite rigorous. He completes the section with a few theorems about finite and infinite sets.

¶64. *Definition.* A set S is said to be *infinite* when it is similar to a proper subset of itself, otherwise it is said to be *finite.* Dedekind's footnote to this definition contains some important historical notes.

> In this form I submitted the definition of the infinite which forms the core of my whole investigation in September, 1882, to G. Cantor and several years earlier to Schwarz and Weber. All other attempts that have come to my knowledge to distinguish the infinite from the finite seem to me to have met with so little success that I think I may be permitted to forego any criticism of them.

¶65. *Theorem.* Every set consisting of a single element is finite.

Proof. A set with a single element has no proper subset. See [2,6]. Q.E.D.

The only proper subset that a singleton set can have is empty, but Dedekind does not accept the empty set. If he did, then he would have restated the theorem to say an empty set is finite. It's the base case that goes along with the theorem in paragraph 70 which says if a set is finite and an element is appended to it, then the resulting set is also finite.

¶66. *Theorem.* There exist infinite sets.

Dedekind's so-called proof of this statement leaves much to be desired. It's worth while quoting, but as it has no mathematical merit whatsoever, it's not worth discussing. The statement should be an axiom, not a theorem. Here's his proof.

> My own realm of thoughts, i.e., the totality S of all things, which can be objects of my thought, is infinite. For if s signifies an element of S, then is the thought s', that s can be object of my thought, itself an element of S. If we regard this as transform $\phi(s)$ of the element s then has the transformation ϕ of S, thus determined, the property that the transform S' is part of S; and S' is certainly proper part of S, because there are elements in S (e.g., my own ego) which are different from such thought s' and therefore are not contained in S'. Finally it is clear that if a, b are different elements of S, their transforms a', b' are also different, that therefore the transformation ϕ is a distinct (similar) transformation [26]. Hence S is infinite, which was to be proved.

Dedekind notes his proof is similar to one of Bolzano's in a footnote: "A similar consideration is found in §13 of the *Paradoxien des Unendlichen* by Bolzano (Leipzig, 1851)." I haven't examined what Bolzano wrote, so I don't know how similar it was.

¶67. *Theorem.* If $R \sim S$, then R is finite or infinite as S is.

Proof. Suppose S is infinite. Then S is similar to a proper subset S' of itself. Since $R \sim S$, by [33], S' is similar to a proper part of R, which, by [33], is similar to R. Therefore, R is similar to a proper subset of itself, so it is infinite by definition [64] Q.E.D.

¶68. *Theorem.* Every set with an infinite subset is itself infinite; every subset of a finite set is finite.

Proof. Let S be a set with an infinite subset T. Then there is a one-to-one function $\psi : T \to T'$ where T' is a proper subset of T. Extend ψ to a function ϕ on S as follows:

$$\phi(s) = \begin{cases} \psi(s) & \text{if } s \in T, \\ s & \text{if } s \notin T. \end{cases}$$

Then ϕ is a one-to-one function as can be seen by examining the cases whether each of two elements lie in T or don't. Also, since T' is a proper subset of T, there is an element t in T but not in T'. Clearly, t is not in the image of ϕ. Therefore, $\phi(S)$ is a proper subset of S, and so S is an infinite set. Q.E.D.

Dedekind supplies the details to this proof that I left out.

¶69. *Theorem.* Every set that is similar to a subset of a finite set is finite. [67,68]

¶70. *Theorem.* If a set S is the union of a finite subset T and an element a not in T, then S is also finite.

Proof. Let $\phi : S \to S$ be a one-to-one function with image $S' = \phi(S)$. By [64], the definition of finite sets, we have to show that $S' = S$.

Now $S = \{a\} \cup T$, so by [23] $S' = \{a'\} \cup T'$. Since ϕ is a one-to-one function, and $a \notin T$, therefore $a' \notin T'$.

We have two cases to consider: $a \notin T'$ and $a \in T'$.

Case 1. Suppose $a \notin T'$. Then $a' = a$ and $T' \subseteq T$. Since T is a finite set and ϕ is a one-to-one function such that $\phi(T) \subseteq T$, therefore $\phi(T) = T$, that is, $T' = T$. Therefore $\phi(S) = \phi(\{a\} \cup T) = \{a'\} \cup T' = \{a\} \cup T = S$. So in this case, $S' = S$ as required.

Case 2. Suppose $a \in T'$. Then $a = b'$ for some $b \in T$. Let U be the subset of T of all elements except b. Then $T = \{b\} \cup U$, and, by [15], $S = \{a\} \cup \{b\} \cup U$. Hence, $S' = \{a'\} \cup \{b'\} \cup U' = \{a'\} \cup \{a\} \cup U$. Define a new function $\psi : T \to T$ by

$$\psi(t) = \begin{cases} a' & \text{if } t = b, \\ \psi(t) & \text{if } t \in U. \end{cases}$$

Then ψ is a one-to-one function. Since T is finite, therefore $\psi(T) = T$. By the definition of ψ, $T = \{a'\} \cup U'$. Hence, $\{a\} \cup T = \{a\} \cup \{a'\} \cup U'$. But $S = \{a\} \cup T$ and $S' = \{a'\} \cup \{a\} \cup U$, therefore $S = S'$, as required.

Thus, in all cases, $S = S'$. Therefore, S is a finite set. Q.E.D.

Dedekind supplies in his proof a couple of details that I left out.

VI. Simply infinite sets. Natural numbers.

¶¶71–80. Dedekind did the general theory of chains in the previous two sections. Now he defines a simply infinite set, that is a set $\mathbf{N}$ along with a one-to-one function $\phi : \mathbf{N} \to \mathbf{N}$ which is the chain of one of its elements, denoted 1. Such a set is enough to model the natural numbers.

¶71. *Definition.* A set $\mathbf{N}$ is said to be *simply infinite* when there exists a one-to-one function $\phi : \mathbf{N} \to \mathbf{N}$ such that $\mathbf{N}$ is the chain, see [44], of an element not contained in the image $\phi(\mathbf{N})$. We will call this element called the *base element* and denote it 1. The set simply infinite set $\mathbf{N}$ is said to be *ordered* by ϕ. As usual $\phi(n)$ will also be denoted n'.

Such a simply infinite set $\mathbf{N}$ is characterized by an element 1 and a transformation $\phi : \mathbf{N} \to \mathbf{N}$ satisfying the following four conditions:

α: $\mathbf{N}' \subseteq \mathbf{N}$.

β: $\mathbf{N} = 1_0$, that is, the chain of 1 is $\mathbf{N}$.

γ: $1 \notin \mathbf{N}'$.

δ: The function ϕ is one-to-one.

From α, β, and δ, it follows that $\mathbf{N}$ is an infinite set by the definition [64].

The characterization of $\mathbf{N}$ given here by Dedekind has become to be known as the Dedekind/Peano axioms for the natural numbers. Peano stated these as axioms, as Dedekind did, as a definition within a larger theory of sets.

¶72. *Theorem.* In every infinite set S there is a simply infinite subset $\mathbf{N}$.

Proof. According to the definition [64], there is a one-to-one function $\phi : S \to S$ such that the image $S' = \phi(S)$ is a proper subset of S. Hence, there is an element in S which is not in S'. Call it 1. Let $\mathbf{N}$ be the chain of 1, that is, $1_0 = \mathbf{N}$. Then $\mathbf{N}$ along with ϕ satisfies the characterizing conditions for a simply infinite set. Q.E.D.

In the next paragraph, Dedekind will call a simply infinite system (a model of) the natural numbers. This theorem says that every infinite set has a subset that is (a model of) the natural numbers.

¶73. *Definition.* In this paragraph Dedekind defines the natural numbers. Actually, he doesn't care which simply infinite set is chosen to be the natural numbers, and it may be easier for us to interpret his statements as saying that a simply infinite set is a set of natural numbers, or that a simply infinite set is a model of the natural numbers. In any case, here is the paragraph quoted in full, translated by Beman.

> If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions $\alpha, \beta, \gamma, \delta$ in [71] and therefore are always

> the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements (compare [134]), form the first object of the *science of numbers* or *arithmetic*. From the general notions and theorems of IV, about the transformation of a system in itself we obtain immediately the following fundamental laws where $a, b, \ldots, m, n, \ldots$ always denote elements of N, therefore numbers, $A, B, C, \ldots$ parts of N, $a', b', \ldots, m', n', \ldots A', B', C', \ldots$ the corresponding transforms, which are produced by the order-setting transformation ϕ and are always elements or parts of N; the transform n' is also called the number *following* n.

As near as I can tell from the translation, Dedekind means that a single simply infinite set $\mathbf{N}$ ordered by ϕ is enough to build the theory of numbers, so long as the only aspects of the set $\mathbf{N}$ that are mentioned are its structure as a simply infinite set. Dedekind explains a little more in [134].

He could have proved at this point that any two simply infinite sets are isomorphic, but he delays that until [133]. I'll call $\mathbf{N}$ the natural numbers in these notes, even though I shouldn't use "the" until after [133].

The number n' following n is often called the *successor* of n. Indeed, the function ϕ is often called the *successor function*.

For the rest of this section, and in later sections, a particular simply infinite set $\mathbf{N}$ and a successor function ϕ is assumed.

¶74. *Theorem.* Every number n is, by [45], contained in its chain n_0, and by [53], the condition $n \in m_0$ is equivalent to $n_0 \subseteq m_0$.

In this and later paragraphs, n_0 denotes the chain generated by the singleton n defined back in paragraph [44].

¶75. *Theorem.* $n_0' = (n_0)' = (n')_0$. [57]

¶76. *Theorem.* $n_0' \subseteq n_0$. [46]

¶77. *Theorem.* $n_0 = \{n\} \cup n_0'$. [58]

¶78. *Theorem.* $\mathbf{N} = \{1\} \cup \mathbf{N}'$. Therefore, every number other than the base-number 1 is the image of a number. [77,71]

In other words, every number n, except 1, is the successor of some number m, $n = m'$.

¶79. *Theorem.* $\mathbf{N}$ is the only number-chain containing the base-number 1.

Dedekind did not explicitly define what is meant by "number-chain." It's probably meant to be evident from the language, but it isn't. The language leaves two competing possibilities. Either a number-chain is the chain n_0 of some number $n \in \mathbf{N}$, or a number-chain is any chain in the set $\mathbf{N}$. Either would do in the proof of this theorem, but paragraph [87] makes it clear. In it Dedekind proves every number-chain is of the form n_0 for some $n \in \mathbf{N}$. Thus, we can make this definition: a *number-chain* is the chain in the set $\mathbf{N}$.

Here's his proof: Let K be a number-chain containing 1. Then by [47], $1_0 \subseteq K$. But $1_0 = \mathbf{N}$, so $\mathbf{N} \subseteq K$. But, of course, $K \subseteq \mathbf{N}$. Therefore, $\mathbf{N} = K$. Q.E.D.

¶80. *Theorem of mathematical induction.* (Inference from n to n'.) In order to show that a theorem holds for all numbers n of a chain m_0, it is sufficient to show

ρ. that it holds for $n = m$, and

σ. that if it holds for a number n in the chain m_0, then it holds for the successor n'.

Proof. The proof follows from the more general theorem. [59,60] Q.E.D.

Dedekind notes that the most frequently occurring case is where $m = 1$, and in that case $m_0 = \mathbf{N}$.

VII. Greater and less numbers.

¶¶81–118. In this section a particular simply infinite set $\mathbf{N}$ and a successor function ϕ is assumed, and the elements of $\mathbf{N}$ are called numbers. Many of the theorems are proved by mathematical induction [80]. Dedekind defines the binary relations of order, namely $<$, $>$, $\leq$, and $\geq$, and proves their essential properties, such as the law of trichotomy [90]. In [87] he shows every chain in $\mathbf{N}$ is generated by one element, that is, it is of the form n_0. To complement chains, which are final segments of $\mathbf{N}$, in [98] he defines initial segments $Z_n = \{m \,|\, m < n\}$. In section XIV, Dedekind uses these sets Z_n as his representative finite sets.

¶81. *Theorem.* Every number n is not equal its successor n'.

Proof. Proof by mathematical induction [80].

ρ. The theorem is true for $n = 1$ by definition [71].

σ. Suppose the theorem is true for n. Then $p = n'$ is not equal to n. Since $p \neq n$, therefore, by [26, 71], $p' \neq n'$. Thus, the theorem is true for n'. Q.E.D.

¶82. *Theorem.* For any number n, $n' \in n'_0$, but $n \notin n'_0$.

Proof. Of course $n' \in n'_0$ by [74, 75]. The other statement, $n \notin n'_0$, will be proved by mathematical induction [80].

ρ. The theorem is true for $n = 1$ by definition [71].

σ. Suppose the theorem is true for n. Let $p = n'$. Then $n \notin p_0$. Therefore $n \neq q$ for every $q \in p_0$. Since ϕ is one-to-one, therefore $n' \neq q'$ for every $q \in p_0$, that is, for every $q' \in p'_0$. In other words, $n' \notin p'_0$. Thus, the theorem is true for n'. Q.E.D.

¶83. *Theorem.* The image-chain n'_0 is a proper subset of the chain n_0. [76, 74, 82]

¶84. *Theorem.* If $m_0 = n_0$, then $m = n$.

Proof. By [74], $m \in m_0$, and, by [77], $m_0 = n_0 = \{n\} \cup n'_0$.

Suppose that $m \neq n$. Then m would be in the chain n'_0, hence, by [74', $m_0 \subseteq n'_0$, that is, $n_0 \subseteq n'_0$. But that contradicts [83].

Therefore $m = n$. Q.E.D.

¶85. *Theorem.* If n is not in a number-chain K, then $K \subseteq n'_0$.

Proof. By mathematical induction [80].

ρ: By [78] the theorem is true for $n = 1$.

σ: Suppose that the theorem is true for n. Let $p = n'$. Suppose that $p \notin K$. Then by [40], $n \notin K$ also. By hypothesis, $K \subseteq n'_0$. But by [77], $n'_0 = p_0 = \{p\} \cup p'_0$, therefore $K \subseteq p \cup p'_0$. But $p \notin K$, so $K \subseteq p'_0$. Thus the theorem is true for p. Q.E.D.

¶86. *Theorem.* If a number n is not in a number-chain K, but n' is in K, then $K = n'_0$.

Since $n \notin K$¡ then by [85], $K \subseteq n'_0$, and since $N' \subseteq K$, then by [47], $n'_0 \subseteq K$. Hence, $K = n'_0$. Q.E.D.

¶87. *Theorem.* In every number-chain K there exists one, and by [84] only one, number k, whose chain k_0 equals K.

Proof. If 1 is in K, then by [79], $K = N = 1_0$. Otherwise, let Z be the set of all numbers not in K. Since $1 \in Z$, but Z is a proper subset of **N**, then, by [79], Z is not a chain. Therefore, Z' is not a subset of Z. Therefore, there is a number $n \in Z$ such that $n' \notin Z$. Then $n' \in K$. By [86], $K = n'_0$, and hence $k = n'$. Q.E.D.

¶88. *Theorem.* If m and n are distinct numbers, then, by [83,84], one and only one of the chains m_0 and n_0 is a proper subset of the other, and either $n_0 \subseteq m'_0$ or $m_0 \subseteq n'_0$.

Proof. We'll divide the proof into to cases depending on whether n is or is not in m_0.

Suppose that $n \in m_0$. Then by [74], $n_0 \subseteq m_0$, so $m \notin n_0$ (because otherwise, by [74], $m_0 \subseteq n_0$ so that $m_0 = n_0$, and then, by [84], $m = n$). Therefore, by [85], $n_0 \subseteq m'_0$.

Suppose that $n \notin m_0$. Then by [85], $m_0 \subseteq n'_0$. Q.E.D.

¶89. *Definition.* The number m is said to be *less* than the number n, written $m < n$, and at the same time n *greater* than m, written $n > m$, when the condition $n_0 \subseteq m'_0$. By [74], the condition is equivalent to the condition $n \in m'_0$.

¶90. *Theorem.* If m and n are any numbers then exactly one of the three cases λ, μ, or ν occurs:

λ: $m = n$, $n = m$, i.e., $m_0 = n_0$.
μ: $m < n$, $n > m$, i.e., $n_0 \subseteq m'_0$.
ν: $m > n$, $n < m$, i.e., $m_0 \subseteq n'_0$.

Proof. If case λ occurs [84], then neither μ nor ν occurs since, by [83], $n_0 \subseteq n'_0$ never occurs. But if λ does not occur, then, by [88], one and only one of the cases μ or ν occurs. Q.E.D.

Dedekind has now proved the law of trichotomy.

¶91. *Theorem.* $n < n'$.

Proof. The condition for case ν in [90] is fulfilled by $m = n'$. Q.E.D.

¶92. *Definition.* To express that the number m is either equal to n or less than n, hence, by [90], not greater than n, we use the symbols $m \leq n$, and $n \geq m$.

¶93. *Theorem.* The three conditions

$m \leq n, \quad m < n', \quad n_0 \subseteq m_0$

are equivalent conditions.

Proof. If $m \leq n$, then from λ and μ in [90] we have $n_0 \subseteq m_0$, because, by [76], $m'_0 \subseteq m$.

Conversely, if $n_0 \subseteq m_0$, and therefore by [74] $n \subseteq m_0$ also, it follows from $m_0 = m \cup m'_0$ that either $n = m$ or $n \subseteq m'_0$, that is, $n > m$. Thus, $m \leq n$.

Also, by [22, 27, 75], the condition $n_0 \subseteq m_0$ is equivalent to $n'_0 \subseteq m'_0$, which, by μ in [90], is equivalent to $m < n'$.

Thus, all three conditions are equivalent. Q.E.D.

¶94. *Theorem.* The three conditions

$m' \leq n, \quad m' < n', \quad m < n$

are equivalent conditions. [93,90]

¶95. *Theorem.* If $l < m$ and $m \leq n$, or if $l \leq m$ and $m < n$, then $l < m$. If $l \leq m$ and $m \leq n$, then $l \leq n$.

Proof. First, suppose that $l < m$ and $m \leq n$. Then by [89,93] $m_0 \subseteq l'_0$ and $n_0 \subseteq m_0$. It follows by [7] that $n_0 \subseteq l'_0$. Thus, $l < m$.

Next, suppose that $l \leq m$ and $m < n$. Then $m_0 \subseteq l_0$ and $n_0 \subseteq m'_0$. But $m_0 \subseteq l_0$ implies $m'_0 \subseteq l'_0$. Therefore, $n_0 \subseteq l'_0$. Thus, $l < m$.

Finally, suppose that $l \leq m$ and $m \leq n$. Then $m_0 \subseteq l_0$ and $n_0 \subseteq m_0$. Therefore, $n_0 \subseteq l_0$. Thus, $l \leq m$. Q.E.D.

¶96. *Theorem.* In every subset T of $\mathbf{N}$ there exists one and only one *least* number k, i.e., a number k which is less than every other number contained in T. If T consists of a single number, then that number is also the least number in T.

Proof. By [44] T_0 is a chain, therefore, by [87], there exists a number k whose chain k_0 is T_0. We'll show that this k is in T and it's less than every other element of T.

By [45], $T \subseteq T_0$, and by [77] $T_0 = k_0 = \{k\} \cup k'_0$, therefore $T \subseteq \{k\} \cup k'_0$. If k were not in T, then $T \subseteq k'_0$, which implies by [47] that $T_0 \subseteq k'_0$, but then $k \in k'_0$, contradicting [83]. Thus, $k \in T$.

Now, every number $t \in T$ lies in k_0, and if $t \neq k$, then $t \in k'_0$, therefore $t > k$. Thus, k is less than every other number in T. Furthermore, this condition characterizes k, so it is the only least number in T. Q.E.D.

Here's one statement that's easier to state when, as Dedekind does, the empty set is rejected. But it's only a little easier to state, since inserting the word "nonempty" in "In every nonempty subset . . . " does the job when the empty set is accepted.

¶97. *Theorem.* The least number in the chain n_0 is n, and 1 is the least of all numbers. [96]

¶98. *Definition.* If n is any number, then we will denote by Z_n the set of all numbers that are *not greater* than n, and hence *not* in n'_0. The condition $m \in Z_n$ is equivalent, by [92,93], to each of the following conditions

$m \leq n, \quad m < n', \quad n_0 \subseteq m_0.$

This set, $Z_n = \{m \mid m \leq n\}$, is an initial segment of numbers. In section XIV Dedekind shows that every finite set is similar to one of these Z_n's, so they are representatives for all finite sets.

Following this definition, Dedekind gives a number of theorems about these segments. For some of them he gives proofs that I'll omit.

¶99. *Theorem.* $1 \in Z_n$, and $n \in Z_n$. [98]

¶100. *Theorem.* The condition $Z_m \subseteq Z_n$ is equivalent to the conditions in [98]:
$m \in Z_n, \quad m \leq n, \quad m < n', \quad n_0 \subseteq m_0.$ [95,99]

¶101. *Theorem.* The trichotomy conditions in [90] can be extended as follows. If m and n are any numbers then exactly one of the three cases λ, μ, or ν occurs
$\lambda.\ m = n, n = m, Z_m = Z_n.$
$\mu.\ m < n, n > m, Z'_m \subseteq Z_n.$
$\nu.\ m > n, n < m, Z'_n \subseteq Z_m.$ [90,100]

¶102. *Theorem.* $Z_1 = \{1\}$. [99,78,98]

¶103. *Theorem.* $\mathbf{N} = Z_n \cup n'_0$. [98]

¶104. *Theorem.* $\{n\} = Z_n \cap n_0$, that is, n is the only common element of the sets Z_n and n_0.
Proof. Of course, [99,74], n is contained in both sets.
By [77], every element of the chain n_0 other than n is in n'_0, so by [98], not in Z_n. Q.E.D.

¶105. *Theorem.* $n' \notin Z_n$. [91,98]

¶106. *Theorem.* $n < m$ if and only if Z_m is a proper subset of Z_n. [100,99,98]

¶107. *Theorem.* Z_n is a proper subset of $Z_{n'}$. [106,91]

¶108. *Theorem.* $Z_{n'} = Z_n \cup \{n'\}$. [98,97,109]

¶109. *Theorem.* The image $(Z_n)'$ of the set Z_n is a proper subset of $Z_{n'}$. [94,98,99,71]

¶110. *Theorem.* $Z_{n'} = \{1\} \cup (Z_n)'$. [78,98,94,99,109]

¶111. *Definition.* If in a set E of numbers there is an element g which is greater than every other number in E, then g is said to be *the greatest* number in E, and, by [90], there can be at most one such greatest number in E. If E contains only one number, then that number is the greatest number in E.

¶112. *Theorem.* n is the greatest number in Z_n. [98]

¶113. *Theorem.* If there is a greatest number g in a set E of numbers, then $E \subseteq Z_g$. [98]

¶114. *Theorem.* If E is a subset of Z_n, or what amounts to the same thing, there exists a number n such that all numbers in E are less than or equal to n, then there is a greatest number in E.
Proof. The set of all numbers p satisfying the condition $E \subseteq Z_p$—and at least one exists by hypothesis—is a chain since, according to [107] $Z_p \subseteq Z_{p'}$, so $E \subseteq Z_p$ implies $E \subseteq Z_{p'}$. Being a chain, it is, by [87], of the form g_0, and by [96, 97], g is the least of these numbers. We'll show g is the greatest number in E.

Since $E \subseteq Z_g$, therefore,by [98], every number in E is less than or equal to g. We have yet to show that $g \in E$. There are two cases to consider, $g = 1$ and $g \neq 1$.

First, suppose that $g = 1$. By [102], Z_g is the single number 1, and so is E, so $g \in E$.

Next, suppose that $g \neq 1$. By [78] $g = f'$ for some f. By [108], $E \subseteq Z_f \cup \{g\}$. If $g \notin E$, then $E \subseteq Z_f$, but then f is less than g contradicting the minimality of g. Therefore, $g \in E$. Q.E.D.

Thus, any set of numbers that is bounded above has a greatest element.

¶115. *Definition.* If $l \leq m$ and $m \leq n$, we say the number m *lies between* l and n (also between n and l).

¶116. *Theorem.* There exists no number lying between n and n'.

Proof. If $m < n'$, by [93] $m \leq n$, and, by [90], $n < m$ is excluded. Q.E.D.

¶117. *Theorem.* If t is a number in a subset T of $\mathbf{N}$, but not the least number [96] in T, then there exists in T one and only one *next less* number s, i.e., a number s such that $s < t$, and that there exists in T no number lying between s and t. Similarly, if t is not the greatest number [111] in T, there always exists in T one and only one *next greater* number u, i.e., a number u such that $t < u$ and that there exists in T no number lying between t and u. At the same time, t is the next greater number than s in T, and t is the next less number than u in T.

Proof outline. Suppose that t is not the least number in T. Let E be the set of all those numbers of T that are less than t. By [98], $E \subseteq Z_t$, so by [114], there exists in E a greatest number s. It can be shown that this s works.

Suppose that t is not the greatest number in T. Then by [96], there is among all the numbers of T that are greater than t a least number u, and it can be shown that this u works.

¶118. *Theorem.* In $\mathbf{N}$ the number n' is next greater than n, and n next less than n'. [116,117]

VIII. Finite and infinite parts of the natural numbers.

¶¶119–123. In this section Dedekind shows that a subset of $\mathbf{N}$ is finite or simply infinite depending on whether it has a greatest element.

¶119. *Theorem.* Every set Z_n, defined in [98], is finite.

Proof. By induction [80].

ρ. By [65, 102], the theorem is true for $n = 1$.

σ. If Z_n is finite, then, by [108,70], $Z_{n'}$ is also finite. Q.E.D.

¶120. *Theorem.* If $m \neq n$, then $Z_m \not\sim Z_n$.

Proof. By symmetry [90], we may assume that $m < n$. By [106], Z_m is a proper subset of Z_n. Since by [119], Z_n is finite, so by [64] the proper subset Z_m cannot be similar to Z_n. Q.E.D.

¶121. *Theorem.* Every subset E of the natural numbers $\mathbf{N}$ which has a greatest number [111] is finite. [113, 119, 68]

¶122. *Theorem.* Every subset U of $\mathbf{N}$ which has no greatest number [111] is simply infinite [71].

Proof. For any number u in U, u is not the greatest number in U, therefore, by [90], there is a larger number in U, so, by [117], there is one and only one next greater number than u in U, which we will denote by $\psi(u)$. This defines a function $\psi : U \to U$.

Next, we'll show that function ψ is a one-to-one function. Let u and v be distinct elements of U. We may assume by symmetry [90] that $u < v$. By [177] and the definition of ψ, $\psi(u) \leq v$ and $v < \psi(v)$. Hence, by [95], $\psi(u) < \psi(v)$, therefore, by [90], $\psi(u) \neq \psi(v)$. Thus, ψ is one-to-one.

Next, we'll show that $\psi(U)$ is a proper subset of U. Let u_1 be the least element of the set U, which exists by [96]. Since every number $u \in U$ is greater than or equal to u_1, and since $u < \psi(u)$, then, by [95], $u_1 < \psi(u)$. Therefore, by [90], u_1 does not equal $\psi(u)$ for any $u \in U$. Thus, u_1 is an element of U but not of $\psi(U)$. Thus, $\psi(U)$ is a proper subset of U, which shows that U is an infinite set [64].

Finally, we will show that U is a simply infinite set. The set U is a chain with respect to ψ, and we will show that it's generated by the one element u_1. As described in [44], we denote by $\psi_0(V)$ the chain generated by a subset V of U ordered by the function ψ. We need to show that $U = \psi_0(u_1)$.

First note that $\psi_0(u_1) \subseteq U$, since the image of ϕ is a subset of U.

We have yet to show that $U \subseteq \psi_0(u_1)$. Suppose that there is a number in U but not in $\psi_0(u_1)$. By [96], there is a least such number w. This w cannot be u_1, because, of course by [45], $u_1 \in \psi_0(u_1)$. Then, by [117] there must be a number v in U which is next less than w, and so $w = \psi(v)$. Since $v < w$, therefore $v \in \psi_0(u_1)$. Then, by [55], $\psi(v)$, which is w, is also in $\psi_0(u_1)$, a contradiction. Hence, every number in U lies in $\psi_0(u_1)$.

Therefore, $U \subseteq \psi_0(u_1)$, and so $U = \psi_0(u_1)$ and U is a simply infinite set. Q.E.D.

¶123. *Theorem.* A subset T of the natural numbers $\mathbf{N}$ is finite or simply infinite depending on whether a greater number in T exists or does not exist. [121,122]

Coupling this result with [114], we can say that a set of numbers is either finite or simply infinite depending on whether it's bounded above or not.

IX. Definition by induction of functions defined on the natural numbers.

¶¶124–131. Dedekind shows that definition by induction is a valid way to construct functions whose domain is the natural numbers. He concludes the section with an alternate characterization of the chain of a set. By definition, the chain $\phi_0(A)$ of a set A is the intersection of all the chains containing A. The alternate characterization is that $\phi_0(A)$ is the union of A and all $\phi^n(A)$ where ϕ^n is the n^{th} iterate of ϕ.

The next theorem is a lemma used in the major theorem which follows it.

¶124. The usual notation developed since section VI is retained, including $\mathbf{N}$ for the natural numbers, while Ω denotes an arbitrary set, not necessarily a set of numbers.

¶125. *Theorem.* Given a function $\theta : \Omega \to \Omega$ and an element $\omega \in \Omega$, there exists a unique function ψ_n such that

I.: $\psi_n : Z_n \to \Omega$.

II.: $\psi_n(1) = \omega$.

III.: $\psi_n(t') = (\theta \circ \psi_n)(t)$ for $t < n$.

Proof by induction. Dedekind notes that condition I is actually redundant being implied by conditions II and III.

ρ. Let $n = 1$. Then, by [102], Z_n consists of the single number 1. The condition $\psi_1(1) = \omega$ defines the function ψ_1 and satisfies the requirements.

σ. Assume that the theorem is true for n. We will show that it is true for $p = n'$. If there were a function $\psi_p : Z_p \to \Omega$ satisfying the conditions II and III for $t < p$, then its restriction to $Z_n \to \Omega$ would satisfy conditions II and III for $t < n$, and so would be uniquely determined on Z_n. Since, by [105,108], p is the only number of the set Z_p not in Z_n, and III also requires $\psi_p(p) = (\theta \circ \psi_p)(n)$, therefore $\psi_p(p) = (\theta \circ \psi_p)(n)$ defines the value of the function $\psi_1(p)$ and satisfies the requirements. Q.E.D.

¶126. *Theorem of definition by induction.* Given a function $\theta : \Omega \to \Omega$ and an element $\omega \in \Omega$, there exists a unique function ψ such that

I.: $\psi : \mathbf{N} \to \Omega$.

II.: $\psi(1) = \omega$.

III.: $\psi(t') = (\theta \circ \psi)(t)$ for all $t \in \Omega$.

Proof. Such a function $\psi : \mathbf{N} \to \Omega$, when restricted to Z_n to give a function $\psi_n : Z_n \to \Omega$ would satisfy the three conditions of [125], and, so, the restricted function would exist and be unique. In particular, $\psi(n) = \psi_n(n)$ is required and determines a function ψ defined on $\mathbf{N}$. Furthermore, this function ψ satisfies conditions I, II, and III since the conditions I, II, and III in [125] are consistent. Q.E.D.

¶127. *Theorem.* Under the hypotheses of the previous theorem, $\psi(T') = (\theta \circ \psi)(T)$ for any subset T of $\mathbf{N}$.

Proof. For $t \in T$, by [126], $\psi(t') = (\theta \circ \psi)(t)$. Q.E.D.

¶128. *Theorem.* Under the same hypotheses, denote by θ_0 the chains [44] which correspond to the function $\theta : \Omega \to \Omega$. Then $\psi(\mathbf{N}) = \theta_0(\omega)$.

Dedekind proves each containment, $\subseteq$ and $\supseteq$, using mathematical induction. I'll omit those straightforword proofs.

An alternative proof is to note that this is a special case of $\psi(\phi_0(A)) = \theta_0(\psi(A))$, a generalization of theorem [57] that I mentioned back in that paragraph. Here in [128], A is the singleton $\{1\}$, $\phi_0(A)$ is $\mathbf{N}$, and $\psi(A)$ is $\{\omega\}$.

¶129. *Theorem.* Under the same hypotheses, $\psi(n_0) = \theta_0(\psi(n))$.

Dedekind gives a proof by induction. This is another special case of $\psi(\phi_0(A)) = \theta_0(\psi(A))$ where A is the singleton $\{n\}$, $\phi_0(A) = n_0$, and $\psi(A) = \{\psi(n)\}$.

¶130. *Remark.* Dedekind points out that the construction of [126] that applies to 1 in $\mathbf{N}$ does not generalize to arbitrary ϕ-chains A_0 in sets S. That is, given $\phi : S \to S$, $A \subseteq S$,

$\phi_0(A) = S$, $\theta : \Omega \to \Omega$, and $\psi : A \to \Omega$, the function ψ does not necessarily extend to $\psi : S \to \Omega$ such that $\psi \circ \phi = \theta \circ \psi$.

His example: $S = \{a, b\}$, $A = \{a\}$, $\phi(a) = b$, $\phi(b) = a$, $\Omega = \{\alpha, \beta, \gamma\}$, $\theta(\alpha) = \beta$, $\theta(\beta) = \gamma$, $\theta(\gamma) = \alpha$, and $\psi(a) = \alpha$.

Dedekind goes on to say that if ψ can be extended to S, then that extension is unique.

We can interpret this situation in terms of category theory. Fix a set A. An object in the category is a set S and a pair of functions, $A \xrightarrow{\zeta} S \xrightarrow{\phi} S$. A map in the category from $A \xrightarrow{\zeta} S \xrightarrow{\phi} S$ to $A \xrightarrow{\eta} \Omega \xrightarrow{\theta} \Omega$ is a function $\psi : S \to \Omega$ such that $\psi \circ \eta = \zeta$ and $\psi \circ \phi = \theta \circ \psi$, in other words, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\zeta} & S & \xrightarrow{\phi} & S \\ \downarrow^{1_A} & & \downarrow^{\psi} & & \downarrow^{\psi} \\ A & \xrightarrow{\eta} & \Omega & \xrightarrow{\theta} & \Omega \end{array}$$

Where the function $A \to A$ is the identity function, often denoted 1_A.

Dedekind has shown that $1 \to \mathbf{N} \to \mathbf{N}$ is the initial object in the category where $A = 1$. That is, there's a unique map in the category from $1 \to \mathbf{N} \to \mathbf{N}$ to any other object $1 \xrightarrow{\eta} \Omega \xrightarrow{\theta} \Omega$. Of course, we can't expect every object $A \xrightarrow{\zeta} S \xrightarrow{\phi} S$ to be the initial object, but for any set A, there is an initial object, and it's just $A \xrightarrow{(1_A,1)} A \times \mathbf{N} \xrightarrow{1_A \times \phi} A \times \mathbf{N}$.

¶131. In this paragraph Dedekind shows that the chain of a set is the union of the set and all its iterated images. That is, if $A \subseteq S$, and $\omega : S \to S$, then

$$\omega_0(A) = A \cup (\bigcup_{n \in \mathbf{N}} \omega^n(A))$$

where ω^n is to be the n^{th} iterate of ω. But before he can do that he has to define what the n^{th} iterate of ω is.

He starts with a set Ω with a binary operation, written multiplicatively as $\mu\nu$. Fix an element μ. It determines a function $\dot{\mu} : \Omega \to \Omega$ where $\dot{\mu}(\nu) = \mu\nu$. Apply theorem [126] to the element μ and the function $\dot{\mu}$ to get a function $\psi : \mathbf{N} \to \Omega$, and denote $\psi(n)$ as μ^n. Dedekind notes that this notion of exponentiation is completely defined by two conditions: $\omega^1 = \omega$, and $\omega^{n'} = \omega\omega^n$.

He says further that if the binary operation is associative (as composition is), then the following two identities can be proved by induction: $\omega^{n'} = \omega^n\omega$, and $\omega^m\omega^n = \omega^n\omega^m$.

Next, Dedekind applies this to the case where Ω is the set of all functions $S \to S$, where S is a fixed arbitrary set, and the binary operation is composition of functions. Consider one such function $\omega : S \to S$. Then ω^n is defined as above, and it means, as intended, the n^{th} iterate of ω. Let A be a subset of S. Then, using induction and some of his previous theorems, he proves that the chain $\omega_0(A)$ of A is the union of A and all the images $\omega^n(A)$ of the iterates of ω applied to A, that is, the equation at the top of this paragraph.

X. The class of simply infinite sets.

¶¶132–134. In this short section, Dedekind shows that any two simply infinite sets are similar, and together all the simply infinite sets form a similarity class as described in [34].

¶132. *Theorem.* All simply infinite systems are similar to each other.

Proof. We've denoted one simply infinite system $\mathbf{N}$ with base element 1 and function $\phi : \mathbf{N} \to \mathbf{N}$. Let another one be Ω with base element ω and function $\theta : \Omega \to \Omega$.

The proof I'll give here is a little different from Dedekind's. His proof relies on induction, but only applied to elements of $\mathbf{N}$.

By theorem [126], there's a unique function $\psi : \mathbf{N} \to \Omega$ such that $\phi(1) = \omega$ and $\psi \circ \phi = \theta \circ \phi$. By theorem [126] again, but applied to Ω rather than $\mathbf{N}$, there's a unique function $\chi : \Omega \to \mathbf{N}$ such that $\chi(\omega) = 1$ and $\chi \circ \theta = \phi \circ \chi$. The composition one way, $\chi \circ \psi$, satisfies $(\chi \circ \psi)(1) = \chi(\omega) = 1$ and $(\chi \circ \psi) \circ \phi = \phi \circ (\chi \circ \psi)$. Therefore it must be the identity function $1_{\mathbf{N}}$ on $\mathbf{N}$ since, by [126], the identity function is the unique function such that $1_{\mathbf{N}}(1) = 1$ and $1_n \circ \phi = \phi \circ 1_n$. Likewise, the composition the other way, $\psi \circ \chi$, must be the identity function 1_Ω on Ω. Thus, ψ and χ are inverse functions yielding the similarity $\mathbf{N} \sim \Omega$. Q.E.D.

¶133. *Theorem.* Every set which is similar to a simply infinite system is simply infinite.

This theorem follows directly from the definition [71] and could have been stated right after it. Dedekind uses induction to prove it, but that's not necessary.

¶134. *Remark.* According to [132] and [133], all simply infinite sets form a class in the sense of [34]. Dedekind explains how any simply infinite set is a model of the natural numbers. Here are his words.

> At the same time, with reference to [71, 73] it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N set in order by the transformation ϕ, and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement ϕ, possesses perfectly general validity for every other simply infinite system Ω set in order by a transformation θ and its elements ν, and that the passage from N to Ω (e.g., also the translation of an arithmetic theorem from one language into another) is effected by the transformation ψ considered in [132, 133], which changes every element n of N into an element ν of Ω, i.e., into $\psi(n)$. This element ν can be called the nth element of Ω and accordingly the number n is itself the nth number of the number-series N. The same significance which the transformation ϕ possesses for the laws in the domain N, in so far as every element n is followed by a determinate element $\phi(n) = n'$, is found, after the change effected by ψ, to belong to the transformation θ for the same laws in the domain Ω, in so far as the element $\nu = \psi(n)$ arising from the change of n is followed by the element $\theta(\nu) = \psi(n')$ arising from the change of n'; we are therefore justified in saying that by $\psi\phi$ is changed into θ, which is symbolically expressed by $\theta = \psi\phi\psi^{-1}$, $\phi = \psi^{-1}\theta\psi$. By these remarks, as I believe, the definition of the notion of numbers given in [73] is fully justified. We now proceed to further applications of theorem [126].

XI. Addition of numbers.

¶¶135–146. In this section, Dedekind defines the operation of addition on $\mathbf{N}$ and proves its usual properties.

¶135. *Definition.* Addition on $\mathbf{N}$ is characterized by the conditions
$m + 1 = m'$, and
$m + n' = (m + n)'$.

In order to show that these conditions determine a unique function, Dedekind fixes m, defines $\theta : \mathbf{N} \to \mathbf{N}$ by $\theta(n) = n'$, and uses the definition by induction in [126] to construct the unique function $\psi : \mathbf{N} \to \mathbf{N}$ such that $\psi(1) = m'$ and $\psi(n') = (\psi(n))'$. Then, denoting $\psi(n)$ by $m + n$, the two characterizing conditions are satisfied.

The proofs of most of the following theorems are straightforward, so I will omit them.

¶136. *Theorem.* $m' + n = m + n'$. [80, 135]

¶137. *Theorem.* $m' + n = (m + n)'$. [136, 135]

¶138. *Theorem.* $1 + n = n'$. [80,135]

¶139. *Theorem.* $1 + n = n + 1$. [138,135]

¶140. *Theorem.* $m + n = n + m$. [80, 139, 135, 136]

¶141. *Theorem.* $(l + m) = n = l + (m + n)$. [80,135]

¶142. *Theorem.* $m + n > m$. [80, 135, 91, 95]

¶143. *Theorem.* The conditions $m > a$ and $m + n > a + n$ are equivalent. [80, 135, 94]

¶144. *Theorem.* If $m > a$ and $n > b$, then $m + n > a + b$. [143, 140, 95]

¶145. *Theorem.* If $m + n = a + n$, then $m = a$. [90, 143]

¶146. *Theorem.* If $l > n$, then there exists one and, by [157], only one number m which satisfies the condition $m + n = l$.

Proof by induction [80].

ρ. Let $n = 1$. If $l > 1$, then, by [89], $l \in \mathbf{N}'$, so $l = m'$ for some number m. Then, by definition [135], $l = m + 1$.

σ. Suppose the theorem is true for n; to show it's true for n'. If $l > n'$, then by [91] and [95], $l > n$ also, hence there is a number k which satisfies the condition $l = k + n$. Since, by [138], $k \neq 1$ (otherwise $l = n'$), then, by [78], $k = m'$ of a number m. Therefore, $l = m' + n$. By [136], $l = m + n'$. Q.E.D.

This theorem allows subtraction $l - n = m$ to be defined when $l > n$. Dedekind doesn't do anything with subtraction, so he doesn't bother to define it here.

XII. Multiplication of numbers.

¶¶147–154. Dedekind defines the operation of multiplication by induction and proves its basic properties.

¶147. *Definition.* Multiplication is defined inductively by

$m1 = m.$

$mn' = mn + m.$

Using definition by induction [126] with $\theta(n) = n + m$, there is a unique function $\psi : \mathbf{N} \to \mathbf{N}$ such that $\psi(1) = m$ and $\psi(n') = \theta(\psi(n))$. Let mn denote $\psi(n)$ and the two conditions are satisfied.

¶148. *Theorem.* $m'n = mn + n$. [80, 147, 135, 141, 140, 136]

¶149. *Theorem.* $1n = n$. [80, 147, 135]

¶150. *Theorem.* $mn = nm$. [80, 147, 149, 148]

¶151. *Theorem.* $l(m + n) = lm + ln$. [80, 135, 147, 135, 141]

¶152. *Theorem.* $(m + n)l = ml + nl$. [151, 150]

¶153. *Theorem.* $(lm)n = l(mn)$. [80, 147, 151]

¶154. *Remark.* Dedekind mentions a minor modification of [147]. Change the requirement to $\psi(1) = k$ in place of $\psi(1) = m$, where k is an arbitrary number. Then with $\theta(n) = n + m$ as before, there is a unique function $\psi : \mathbf{N} \to \mathbf{N}$ such that $\psi(1) = k$ and $\psi(n') = \theta(\psi(n))$. This function ψ would have the value at n' of $\psi(n') = mn + k$.

I don't know what the point of this remark is.

XIII. Exponentiation of numbers.

¶¶155–158. Dedekind defines exponentiation of numbers an proves their basic properties.

¶155. *Definition.* Exponentiation is determined by the two conditions

$a^1 = a$, and

$a^{n'} = a\, a^n.$

Of course, the constructed function ψ that defines exponentiation by $a^n = \psi(n)$ satisfies $\psi(1) = a$ and $\psi(n') = \theta(\psi(n))$ where $\theta(n) = an$.

¶156. *Theorem.* $a^{m+n} = a^m a^n$. [80, 135, 155, 153]

¶157. *Theorem.* $(a^m)^n = a^{mn}$. [80, 155, 147, 156]

¶158. *Theorem.* $(ab)^n = a^n b^n$. [80, 155, 150, 153]

XIV. The number of elements in a finite set.

¶¶159-172. Dedekind shows that each finite set is similar to exactly one of the sets Z_n defined in [98], and calls n the number of elements of that finite set. He begins the study of finite cardinal numbers, showing, for example [170], that a finite union of finite sets is finite.

¶159. *Theorem.* A set Σ is infinite if and only if each of the number sets Z_n defined in [98] is similar to a subset of Σ.

Proof. If Σ is infinite, then by [72] there is a subset T of Σ which is simply infinite, which, by [132], is similar to the natural numbers **N**. So, by [35], every set Z_n which is a subset of **N** is similar to a subest of T, so is similar to a subset of Σ.

Dedekind mentions that the proof of the converse is more complicated. The problem is to piece together a one-to-one function from Σ to a proper subset of itself.

Assume that each Z_n is similar to a subset of Σ. That means that there is are one-to-one functions $\alpha_n : Z_n \to \Sigma$. The first part of this proof, as Dedekind says, is to replace the functions α_n by functions ψ_n so that each is an extension of the previous, that is, if $m \leq n$, then ψ_n agrees with ψ_m on Z_m.

Dedekind's method is a sort of higher-order approach. Let Ω be the set of one-to-one functions $Z_n \to \Sigma$ for various n. Dedekind defines a function $\theta : \Omega \to \Omega$ as follows. Note that the various α_n are all elements of Ω. Let β be an element of Ω. Then $\beta : Z_n \to \Sigma$ for some n. The set $a_{n'}(Z_{n'})$ is not a subset of $\beta(Z_n)$, for if it were, then $Z_{n'}$ would be similar, by [35], to a subset of Z_n, hence, by [107], to a proper subset of itself, and consequently infinite, contradicting [119]. Therefore, there exists one or more numbers p in $Z_{n'}$ such that $\alpha_{n'}(p)$ is not an element of $\beta(Z_n)$. From among these elements p select the least, k, which exists by [96]. Since, by [108], $Z_{n'}$ is the union of Z_n and n', define a function $\gamma : Z_{n'} \to \Sigma$ so that $\gamma(m) = \beta(m)$ for $m \in Z_n$, and $\gamma(n') = \alpha_{n'}(k)$. This function γ is a one-to-one function, so we let $\theta(\beta) = \gamma$ in order to define the function $\theta : \Omega \to \Omega$.

Now apply [126] with the element $\omega \in \Omega$ being α_1. Then there is a function $\psi : \mathbf{N} \to \Omega$, which, when we denote $\phi(n)$ as ϕ_n, satisfies the conditions

$\psi_1 = \alpha_1$, and

$\psi_{n'} = \theta(\psi_n)$.

Next, we'll show by induction that ψ_n is a one-to-one function from Z_n to Σ.

ρ. The function ψ_1 is a one-to-one function since α_1 is.

σ. Assume that ψ_n is a one-to-one function. Then, by the construction of $\psi_{n'} = \theta(\psi_n)$, $\psi_{n'}$ is also a one-to-one function.

A similar induction shows that $\psi_n(m) = \psi_m(m)$ for $m \leq n$. (I'll omit that straightforward proof.)

Now, define a function $\chi : \mathbf{N} \to \Omega$ by $\chi(n) = \psi_n(n)$. Then, by [21], χ extends each function ψ_n (details omitted). Furthermore, χ is a one-to-one function (details again omitted). Thus, by [71, 67], Ω contains a simple infinite subset $\chi(\mathbf{N})$. Therefore, by [68], Ω is also an infinite set. Q.E.D.

¶160. *Theorem.* A set Σ is finite or infinite according to whether or not some Z_n is similar to it.

Proof. Suppose that Σ is finite. Then, by [159], there is a set Z_k not similar to Σ, and, by [96], there is a least k such that $Z_k \not\sim \Sigma$. As Σ is not empty (as Dedekind excludes the empty set), therefore $k \neq 1$, so $k = n'$ for some number n. Since $n < k$, there is a one-to-one function $\psi : X_n \to \Sigma$. If $\psi(Z_n)$ were a proper subset of Σ, then thee would be an element $z \notin \psi(Z_n)$, and, consequently, ψ could be extended to a one-to-one function $Z_{n'} \to \Sigma$, which it can't since $Z_k \not\sim \Sigma$. Thus, $\psi(Z_n) = \Sigma$, so $\Sigma \sim Z_n$.

Suppose, now, that Σ is similar to some X_n. Then by [119,67], Σ is finite. Q.E.D.

¶161. *Definition.* If a set Σ is finite, then by [160] there is one, and by [120, 33], only one single number n for which a $Z_n \sim \Sigma$. Call this number n the *number [Anzahl] of elements* of Σ. In this sense, numbers are called *cardinal numbers.* When a one-to-one correspondence $\psi : Z_n \to \Sigma$ is under consideration, the various elements of Σ may be denoted a_m, where $m \in Z_n$ and $a_m = \psi(m)$. In this way, the elements of Σ are *ordered* by ψ, and the numbers m are ordinal numbers.

Dedekind does not have a symbol for the number of elements of a set Σ, but I'll use the notation $|\Sigma|$ for that.

¶162. *Theorem.* All sets similar to a finite set have the same number of elements. [33, 161]

¶163. *Theorem.* The numbers which are less than or equal to n, is n, that is, $|Z_n| = n$. [32]

¶164. *Theorem.* The number of elements in a singleton set is 1, and, conversely, a set whose number of elements is 1 is a singleton set. [2, 26, 32, 102, 161]

¶165. *Theorem.* If T is a proper subset of a finite set Σ, then $|T| < |\Sigma|$.

Proof. By [68], T is a finite set, so similar to a set Z_m where $m = |T|$. Let $n = |\Sigma|$ so that $Z_n \sim \Sigma$. By [35], T is similar to a proper subset E of Z_n, and, by [33], $Z_m \sim E$. Now, if $n \leq m$, then $Z_n \subseteq Z_m$, and, by [7], E would be a proper subset of Z_m, which would imply that Z_m is infinite, which contradicts [119]. Therefore, by [90], $m < n$. Q.E.D.

¶166. *Theorem.* If $\Gamma = B \cup \{\gamma\}$ where $\gamma \notin B$, and $|B| = n$, then $|\Gamma| = n'$.

Proof. Let $\psi : Z_n \to B$ be a one-to-one correspondence. By [105, 108], ψ can be extended to a one-to-one correspondence $Z_{n'} \to \Gamma$. Q.E.D.

¶167. *Theorem.* If γ is an element of a set Γ with n' elements, then n is the number of all other elements of Γ. [166, 26]

¶168. *Theorem.* If A and B are two finite sets with no elements in common, then $|A \cup B| = |A| + |B|$.

Proof. Let $|A| = m$ and $|B| = n$. Proof by induction [80].

ρ: The theorem is true for $n = 1$ by [166, 164, 135].

σ: Suppose that the theorem is true for n. To show that it's true for n'. Let Γ be a set with n' elements. By [167], $\Gamma = B \cup \{\gamma\}$ where γ is one element of Γ and B the set of the other n elements of Γ. Let $\Sigma = A \cup B$. Then by hypothesis, $|\Sigma| = m + n$. Therefore, by [166], $|\Sigma \cup \{\gamma\}| = (m+n)'$, which, by [135], equals $m + n'$. By [15], $\Sigma \cup \{\gamma\} = A \cup B \cup \{\gamma\} = A \cup \Gamma$, so $|A \cup \Gamma| = m + n'$. Q.E.D.

¶169. *Theorem.* If A and B are two finite sets, then $|A \cup B| \leq |A| + |B|$.

Proof. Let $|A| = m$ and $|B| = n$. If $B \subseteq A$, then $A \cup B = A$, so, by [142], $|A \cup B| = |A| \leq |A| + |B|$. But if B is not a subset of A, let T be the set of elements in B that are not in A, then, by [165], $|T| = p \leq n$, so, by [143], $|A \cup B| = |A \cup T| = m + p \leq m + n$. Q.E.D.

Note that Dedekind needs the first part of the proof because he does not accept the empty set.

¶170. *Theorem.* Every set that is the union of n finite sets is a finite set.

Proof. By induction [80].

ρ. By [8], the theorem is true for $n = 1$.

σ. Suppose that the theorem is true for n. Let Σ be the union of n' finite sets. Let A be one of those sets, and let B be the union of the rest of those sets. By [167], B is the union of n finite sets, therefore B is a finite set. But $\Sigma = A \cup B$, so, by [169], Σ is a finite set. Q.E.D.

Note that to say that there are n sets requires that those sets be elements of another set of n elements, since only a set can have a number of elements. Dedekind has declared back in [2] that sets may be elements of other sets.

¶171. *Theorem.* If ψ a non-one-to-one function defined on a finite set Σ of n elements, then $|\psi(\Sigma)| < n$.

Proof. For each element in $\upsilon \in \psi(\Sigma)$ choose one element $\tau \in \Sigma$ such that $\psi(\tau) = \upsilon$. Let T be the subset of Σ of all these chosen elements τ. Then ψ is one-to-one on T. By [26], T is a proper subset of Σ. Then, by [162, 165], $|\psi(\Sigma)| = |\psi(T)| = |T| < |\Sigma| = n$. Q.E.D.

¶172. *Final remark.* Dedekind's final remark concerns what we would call multisets, that is, sets where each element has a multiplicity. He notes that for the function ψ in the last theorem, we could consider the image $\psi(\Sigma)$ to be a multiset where the multiplicity of an element υ is the number of its preimages τ, $\psi(\tau) = \upsilon$. Then, as a multiset, $|\psi(\Sigma)| = |\Sigma|$.

He says this notion is useful in many ways, calls it an extension of the original notion of sets, and concludes "but it does not lie in the line of this memoir to go further into their discussion."

Preface to the first edition

[I include below Beman's translation of Dedekind's preface to the first edition. It has Dedekind's philosophy of numbers that underlies his mathematics. In several places it refers to his earlier work *Continuity and Irrational Numbers*, and a fair amount of the preface is devoted to Dedekind's priority of the concepts he developed for that work. The footnotes are from the preface.]

In science nothing capable of proof ought to be accepted without proof. Though this demand seems so reasonable yet I cannot regard it as having been met even in the most recent methods of laying the foundations for the simplest science; viz., that part of logic which deals with the theory of numbers.[1] In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. My answer to the problems propounded in the title of this paper is, then, briefly this: numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things. It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number-domain that we are prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind.[2] If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensable foundation, as I have already affirmed in an announcement of this paper,[3] must, in my judgment, the whole science of numbers be established. The design of such a presentation I had formed before the publication of my paper on *Continuity*, but only after its appearance and with many interruptions occasioned by increased official duties and other necessary labors, was I able in the years 1872 to 1878 to commit to paper a first rough draft which several mathematicians examined and partial1y discussed with me. It bears the same title and contains, though not in the best order, all the essential fundamental ideas of my present paper, in which they are more carefu1ly elaborated, As such main points I mention here the sharp distinction between finite and infinite [64], the notion of the number [*Anzahl*] of things [161], the proof that the form of argument known as complete induction (or the inference from n to $n + 1$) is really conclusive [59, 60, 80], and that therefore the definition by induction (or recursion) is determinate and consistent [126],

This memoir can be understood by anyone possessing what is usually called good common sense; no technical, philosophic, or mathematical knowledge is in the least degree required. But I feel conscious that many a reader will scarcely recognize in the shadowy forms which I bring before him his numbers which all his life long have accompanied him as faithful and familiar friends; he will be frightened by the long series of simple inferences corresponding to our step-by-step understanding, by the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend, and will become impatient at being compelled to follow out proofs for truths which to his supposed inner consciousness seem at once evident and certain. On the contrary in just this possibility of reducing such truths to others more simple, no matter how long and apparently artificial the series of inferences, I recognize a convincing proof that their possession or belief in them is never given by inner consciousness but is always gained only by a more or less complete repetition of the individual inferences. I like to compare this action of thought, so difficult to trace on account of the rapidity of its

[1]Of the works which have come under my observation I mention the valuable *Lehrbuch der Arithmetik und Algebra* of E. Schröder (Leipzig, 1873), which contains a bibliography of the subject, and in addition the memoires of Kronecker and von Helmholtz upon the Number-Concept and upon Counting and Measuring (in the collection of philosophical essays published in honor of E. Zeller, Leipzig, 1887). The appearance of these memoirs has induced me to publish my own views in many respect similar but in foundation essentially different, which I formulated many years ago in absolute independence of the works of others.

[2]See Section III of my memoir *Stetigkeit un irrationale Zahlen* [*Continuity and Irrational Numbers*], (Braunschweig, 1872).

[3]Dirichlet's *Vorlesungen über Zahlentheorie*, third edition, 1879, §163, note on page 470.

performance, with the action which an accomplished reader performs in reading; this reading always remains a more or less complete repetition of the individual steps which the beginner has to take in his wearisome spelling-out; a very small part of the same, and therefore a very small effort or exertion of the mind, is sufficient for the practiced reader to recognize the correct, true word, only with very great probability, to be sure; for, as is well known, it occasionally happens that even the most practiced proof-reader allows a typographical error to escape him, i.e., reads falsely, a thing which would be impossible if the chain of thoughts associated with spelling were fully repeated. So from the time of birth, continually and in increasing measure we are led to relate things to things and thus to use that faculty of the mind on which the creation of numbers depends; by this practice continually occurring, though without definite purpose, in our earliest years and by the attending formation of judgments and chains of reasoning we acquire a store of real arithmetic truths to which our first teachers later refer as to something simple, self-evident, given in the inner consciousness; and so it happens that many very complicated notions (as for example that the number [*Anzahl*] of things) are erroneously regarded as simple. In this sense which I wish to express by the word formed after a well-known saying [phrase in Greek], I hope that the following pages, as an attempt to establish the science of numbers upon a uniform foundation will find a generous welcome and that other mathematicians will be led to reduce the long series of inferences to more moderate and attractive proportions.

In accordance with the purpose of this memoir I restrict myself to the consideration of the series of so-called natural numbers. In what way the gradual extension of the number-concept, the creation of zero, negative, fractional, irrational and complex numbers are to be accomplished by reduction to the earlier notions and that without any introduction of foreign conceptions (such as that of measurable magnitudes, which according to my view can attain perfect clearness only through the science of numbers), this I have shown at least for irrational numbers in my former memoir on *Continuity* (1872); in a way wholly similar, as I have already shown in Section III of that memoir, may the other extensions be treated, and I propose sometime to present this whole subject in systematic form. From just this point of view it appears as something self-evident and not new that every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers,—a declaration I have heard repeatedly from the lips of Dirichlet. But I see nothing meritorious—and this was just as far from Dirichlet's thought—in actually performing this wearisome circumlocution and insisting on the use and recognition of no other than rational numbers. On the contrary, the greatest and most fruitful advances in mathematics and other sciences have invariably been made by the creation and introduction of new concepts, rendered necessary by the frequent recurrence of complex phenomena which could be controlled by the old notions only with difficulty. On this subject I gave a lecture before the philosophic faculty in the summer of 1854 on the occasion of my admission as privat-docent in Gottingen. The scope of this lecture met with the approval of Gauss; but this is not the place to go into further detail.

Instead of this I will use the opportunity to make some remarks relating to my earlier work, mentioned above, on *Continuity and Irrational Numbers*. The theory of irrational numbers there presented, wrought out in the fall of 1853, is based on the phenomenon (Section IV) occurring in the domain of rational numbers which I designate by the term cut [*Schnitt*] and which I was the first to investigate carefully; it culminates in the proof of the continuity of the new domain of real numbers (Section V, iv.). It appears to me to be somewhat simpler, I might say easier, than the two theories, different from it and from each other, which have been proposed by Weierstrass and G. Cantor, and which likewise are perfectly rigorous. It has since been adopted without essential modification by U. Dini in his *Fondamenti per la teorica delle funzioni di variabili reali* (Pisa, 1878); but the fact that in the course of this exposition my name happens to be mentioned, not in the description of the purely arithmetic phenomenon of the cut but when the author discusses the existence of a measurable quantity corresponding to the cut, might easily lead to the supposition that my theory rests upon the consideration of such quantities. Nothing could be further from the truth; rather have I in Section III of my paper advanced several reasons why I wholly reject the introduction of measurable quantities; indeed, at the end of the paper I have pointed out with respect to their existence that for a great part of the science of space the continuity of its configurations is not even a necessary condition, quite aside from the fact that in works on geometry, arithmetic is only casually mentioned by name but is never clearly defined and therefore cannot be employed in demonstrations. To explain this matter more clearly I note the following example: If we select three non-collinear points A, B, and C at pleasure, with the single limitation that the ratios of the distances AB, AC, and BC are algebraic numbers,[4] and regard as existing in space only those points M for which the ratios of AM, BM, and CM to AB are likewise algebraic numbers, then the space made up of such points M is, as is easy to see, everywhere discontinuous; but in spite of this discontinuity, and despite the existence of gaps in this space, all constructions that occur in Euclid's *Elements*, can, so far as I can see, be just as accurately effected as in perfectly continuous space; the discontinuity of this space would not be noticed in Euclid's science, would not be felt at all. If any one should say that we cannot conceive of space as anything else than continuous, I should venture to doubt it and to call attention to the fact that a far advance, refined scientific training is demanded in order to perceive clearly the essence of continuity and to comprehend that besides rational quantitative relations, also irrational, and besides algebraic, also transcendental quantitative relations are conceivable. All the more beautiful it appears to me that without any notion of measurable quantities and simply by a finite system of simple thought-steps man can advance to the creation of the pure continuous number-domain; and only by this means in my view is it possible for him to render the notion of continuous space clear and definite.

The same theory of irrational numbers founded upon the phenomenon of the cut is set forth in the *Introduction à la théorie des fonctions d'une variable* by J. Tannery (Paris, 1886). If I rightly understand a passage in the preface to this work, the author has thought out his theory independently, that is, at a time when not only my paper, but Dini's *Fondamenti* mentioned in the same preface, was unknown to him. This agreement seems to me a gratifying proof that my conception conforms to the nature of the case, a fact recognized by other mathematicians, e.g., by Pasch in his *Einleitung in die Differential- und Integral-rechnung* (Leipzig, 1883). But I cannot quite agree with Tannery when he calls this theory the development of an idea due to J. Bertrand and contained in his *Traité d'arithmétique*, consisting in this that an irrational number is defined by the specification of all rational numbers that are less and all those that are greater than the number to be defined. As regards this statement which is repeated by Stolz—apparently without careful investigation—in the preface to the second part of his *Vorlesungen über allgemeine Arithmetik* (Leipzig, 1886), I venture to remark the following: That an irrational number is to be considered as fully defined by the specification

[4]Dirichlet's *Vorlesungen über Zahlentheorie*, §159 of the second edition, §160 of the third.

just described, this conviction certainly long before the time of Bertrand was the common property of all mathematicians who concerned themselves with the notion of the irrational. Just this manner of determining it is in the mind of every computer who calculates the irrational root of an equation by approximation, and if, as Bertrand does exclusively in his book (the eighth edition, of the year 1885, lies before me), one regards the irrational number as the ratio of two measurable quantities, then is this manner of determining it already set forth in the clearest possible way in the celebrated definition which Euclid gives of the equality of two ratios (*Elements*, V, 5). This same most ancient conviction has been the source of my theory as well as that of Bertrand and many other more or less complete attempts to lay the foundations for the introduction of irrational numbers into arithmetic. But though one is so far in perfect agreement with Tannery, yet in actual examination he cannot fail to observe that Bertrand's presentation, in which the phenomenon of the cut in its logical purity is not even mentioned, has no similarity whatever to mine, inasmuch as it resorts at once to the existence of a measurable quantity, a notion which for reasons mentioned above I wholly reject. Aside form this fact this method of presentation seems also in the succeeding definitions and proofs, which are based on the postulate of this existence, to present gaps so essential that I still regard the statement made in my paper (Section VI) that the theorem $\sqrt{2}\sqrt{3} = \sqrt{6}$ has nowhere yet been strictly demonstrated, as justified with respect to this work also, so excellent in many other regards and with which I was unacquainted at that time.

R. Dedekind.

Harzburg, October 5, 1887.

Preface to the second edition

The present memoir soon after its appearance met with both favorable and unfavorable criticisms; indeed serious faults were charged against it. I have been unable to convince myself of the justice of these charges, and I now issue a new edition of the memoir, which for some time has been out of print, without change, adding only the following notes to the first preface.

The property which I have employed as the definition of the infinite system had been pointed out before the appearance of my paper by G. Cantor (*Ein Beitrag zur Mannigfaltigkeitslehre, Crelle's Journal*, Vol. 84, 1878), as also by Bolzano (*Paradoxien des Unendlichen* §20, 1851). But neither of these authors made the attempt to use this property for the definition of the infinite and upon this foundation to establish with rigorous logic the science of numbers, and just in this consists the content of my wearisome labor which in all its essentials I had completed several years before the appearance of Cantor's memoir and at a time when the work of Bolzano was unknown to me even by name. For the benefit of those who are interested in and understand the difficulties of such an investigation, I add the following remark. We can lay down an entirely different definition of the finite and infinite, which appears still simpler since the notion of similarity of transformation is not even assumed, viz.:

"A system S is said to be finite when it may be so transformed in itself [36] that no proper part [6] of S is transformed in itself; in the contrary case S is called an infinite system."

Now let us attempt to erect our edifice upon this new foundation! We shall soon meet with serious difficulties, and I believe myself warranted in saying that the proof of the perfect agreement of this definition with the former can be obtained only (and then easily) when we are permitted to assume the series of natural numbers as already developed and to make use of the final considerations in [131]; and yet nothing is said of all these things in either the one definition or the other! From this we can see how very great is the number of steps in thought needed for such a remodeling of a definition.

About a year after the publication of my memoir I became acquainted with G. Frege's *Grundlagen der Arithmetik*, which had already appeared in the year 1884. However different the view of the essence of number adopted in that work is from my own, yet it contains, particularly from §79 on, points of very close contact with my definition [44]. The agreement, to be sure, is not easy to discover on account of the different from of expression; but the positiveness with which the author speaks of the logical inference from n to $n+1$ shows plainly that here he stands upon the same ground with me. In the meantime E. Schröder's *Vorlesungen über die Algebra der Logik* has been almost completed (1890-1891). Upon the importance of this extremely suggestive work, to which I pay my highest tribute, it is impossible here to enter further; I will simply confess that in spite of the remark made on p. 253 of Part I, I have retained my somewhat clumsy symbols [8] and [17]; they make no claim to be adopted generally but are intended simply to serve the purpose of this arithmetic paper to which in my view they are better adapted than sum and product symbols.

R. Dedekind.

Harzburg. August 24, 1893

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