Math 120 Calculus I
Second Test Answers
October 2015

Scale. 98–100 A+, 93–97 A,89–92 A-, 85–88 B+, 81-84 B, 77–80 B-, 70–76 C+, 62–69 C, 55–61 C-, 45–54 D+, 35–44 D. Median 83.

1. [12] On implicit differentiation. The point $(1, 0)$ lies in the curve $y^3 - y = 1.2(x^3 - x)$. Determine the slope of the line tangent to the curve at that point.

Start by taking the derivative of $y^3 - y = 1.2(x^3 - x)$ with respect to x. You'll get

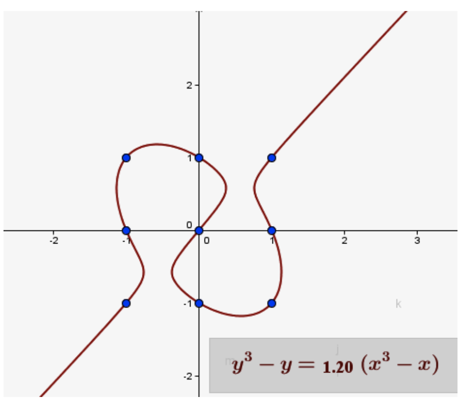
$$3y^2 \frac{dy}{dx} - \frac{dy}{dx} = 1.2(3x^2 - 1).$$

Solve for $\dfrac{dy}{dx}$ from that equation:

$$\frac{dy}{dx} = \frac{1.2(3x^2 - 1)}{(3y^2 - 1)}$$

That's the slope of the tangent line at a point (x, y) on the curve. When (x, y) is the point $(1, 0)$ that is equal to

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{1.2(3 - 1)}{(0 - 1)} = -2.4.$$



2. [12] Recall the definition of derivatives in terms of limits, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x + h) - f(x)}{h}$. Use that definition to show that the derivative of $f(x) = 5 + \dfrac{1}{x}$ is $f'(x) = -\dfrac{1}{x^2}$. (Do not use any of the rules of differentiation, just the definition.)

Here are the steps if you don't leave any of them out. Some can be combined, of course.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(5 + 1/(x + h)) - (5 + 1/x)}{h} \\ &= \lim_{h\to 0} \frac{5 + 1/(x + h) - 5 - 1/x}{h} \\ &= \lim_{h\to 0} \frac{1/(x + h) - 1/x}{h} \\ &= \lim_{h\to 0} \frac{x - (x + h)}{h(x + h)x} \\ &= \lim_{h\to 0} \frac{-h}{h(x + h)x} \\ &= \lim_{h\to 0} \frac{-1}{(x + h)x} \\ &= \frac{-1}{x^2} \end{aligned}$$

3. [12] On logarithmic differentiation. The function $y = f(x) = (x^2 + 4)^{\sqrt{x}}$ cannot be differentiated by the power rule since the exponent is not constant, and it can't be differentiated by the exponential rule since the base is not constant, but you can find its derivative with logarithmic differentiation or by using the identity $a^b = e^{b \ln a}$. Find its derivative. Show your work, and write carefully. Express your answer $f'(x)$ in terms of x.

First take ln of the equation $f(x) = (x^2 + 4)^{\sqrt{x}}$ to get

$$\ln f(x) = \sqrt{x}\, \ln(x^2 + 4).$$

Then differentiate with respect to x. You'll need the product rule and the chain rule.

$$\frac{f'(x)}{f(x)} = \frac{1}{2\sqrt{x}} \ln(x^2 + 4) + \sqrt{x}\, \frac{2x}{x^2 + 4}.$$

Finally solve for $f'(x)$ to get

$$f'(x) = (x^2 + 4)^x \left(\frac{1}{2\sqrt{x}} \ln(x^2 + 4) + \sqrt{x}\, \frac{2x}{x^2 + 4} \right).$$

4. [40; 8 points each part] Differentiate the following functions. Do not simplify your answers. Use parentheses properly.

a. $f(t) = 7t^4 - 4\sqrt{t} + 6 + \dfrac{25}{t^2}$

Use the power rule to find

$$f'(t) = 28t^3 - \frac{2}{\sqrt{t}} - \frac{50}{t^3}.$$

You could also write the derivative as

$$f'(t) = 28t^3 - \tfrac{1}{2} t^{-1/2} - 50t^{-3}.$$

b. $g(x) = \sin 5x + \tan 3x$

You'll need the chain rule to find the derivative of each term.

$$g'(x) = 5\cos 5x + 3\sec^2 3x$$

c. $y = \dfrac{e^x + \ln x}{5 + \sqrt{x}}$

Use the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^x + \ln x)'(5+\sqrt{x}) - (e^x + \ln x)(5+\sqrt{x})'}{(5+\sqrt{x})^2} \\ &= \frac{(e^x + 1/x)(5+\sqrt{x}) - (e^x + \ln x)\dfrac{1}{2\sqrt{x}}}{(5+\sqrt{x})^2} \end{aligned}$$

d. $f(x) = x \arctan x$. (Note that the inverse tangent function $\arctan x$ is often written $\tan^{-1} x$, but it does not equal $(\tan x)^{-1}$.)

Use the product rule.

$$\begin{aligned} f'(x) &= (x)' \arctan + x(\arctan x)' \\ &= \arctan x + \frac{x}{1+x^2} \end{aligned}$$

e. $f(\theta) = \theta^3 \cos\sqrt{\theta}$

Use the product rule and then the chain rule.

$$\begin{aligned} f'(\theta) &= (\theta^3)'(\cos\sqrt{\theta}) + (\theta^3)(\cos\sqrt{\theta})' \\ &= 3\theta^2(\cos\sqrt{\theta}) - \theta^3(\sin\sqrt{\theta})\frac{1}{2\sqrt{\theta}} \end{aligned}$$

5. [12] Consider the function $f(x) = \dfrac{\sqrt{x^2+1}}{10x-2}$.

a. Determine the limit $\lim\limits_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{10x-2} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{10-2/x} = \frac{1}{10}$$

b. Does $f(x)$ have any horizontal asymptotes? If yes, write the equation(s) for the horizontal line(s) that the curve $y = f(x)$ is asymptotic to.

Yes. $y = \frac{1}{10}$. (Also $y = -\frac{1}{10}$ since as $x \to -\infty$, $y \to -\frac{1}{10}$, but not required for credit.)

c. Does $f(x)$ have any vertical asymptotes? If yes, write the equation(s) for the vertical line(s) that the curve $y = f(x)$ is asymptotic to.

Yes, when $10x - 2 = 0$ the denominator is 0, so a vertical asymptote is the line $x = \frac{1}{5}$.

6. [12] A 50 foot ladder is leaning against a tall building. If the base of the ladder is slipping away from the building at 0.5 feet per minute, how fast is the top of the ladder descending when it is 40 feet up from the ground?

Let x denote the distance in feet form the base of the ladder to the building, and y the height in feet of the top of the ladder. Then by the Pythagorean theorem, $x^2 + y^2 = 50^2$. Differentiate that equation with respect to t to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Therefore,

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $y = 40$, $x = 30$, so $\dfrac{dy}{dt} = -\frac{3}{4}\,0.5 = -\frac{3}{8}$. The top of the ladder is descending 3/8 feet per minute.