Math 120 Calculus I
Final Answers
December 2016

Scale. [to be determined]

1. [22] Consider the function \( f(x) = \frac{x}{1 + x^2} \). Its derivative is \( f'(x) = \frac{1 - x^2}{(1 + x^2)^2} \), and its second derivative is \( f''(x) = \frac{4x^3 - 6x}{(1 + x^2)^3} \).

a. [3] What are the \( x \)-intercepts and \( y \)-intercepts of \( f \)?

The only \( x \)-intercept is the origin and the \( y \)-intercept is also the origin. In other words, the graph of \( f \) only crosses either axis at the origin.

b. [3] What are the critical points for \( f \)?

The numerator \( 1 - x^2 \) of the derivative \( f' \) is 0 when \( x = \pm 1 \).

c. [3] What are the inflection points for \( f \)?

The numerator \( 4x^3 - 6x \) of the second derivative \( f'' \) is 0 when \( x = 0, \pm \sqrt{3}/2 \).

d. [4] Are there any vertical asymptotes? Are there any horizontal asymptotes?

There are no vertical asymptotes, but the \( x \)-axis is a horizontal asymptote since \( \lim_{x \to \pm\infty} f(x) = 0 \).

e. [6] Sketch the graph of \( f \). Show intercepts, critical points, inflection points, and asymptotes.

The curve looks like this, but it should be also annotated with the points mentioned above.

\[ \begin{align*}
   f(x) &= \frac{x}{1 + x^2} \\
   f'(x) &= \frac{1 - x^2}{(1 + x^2)^2} \\
   f''(x) &= \frac{4x^3 - 6x}{(1 + x^2)^3} 
\end{align*} \]

2. [15; 5 points each part] On limits. Evaluate each of the following limits in parts a and b if it exists, but if it doesn’t then explain why.

a. \( \lim_{x \to 0} \frac{\sin^2 3x}{5x^2} \)

One way to evaluate this limit is to rewrite it as

\[ \lim_{x \to 0} \frac{\sin 3x \sin 3x}{5x^2} = \frac{9}{5} \]

which equals \( \frac{9}{5} \) since \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \).

You could also use L’Hôpital’s rule since this limit is of the indeterminant form 0/0. Then the limit is equal to

\[ \lim_{x \to 0} \frac{6 \sin 3x \cos 3x}{10x} \]

It’s still of the form 0/0, and another application of L’Hôpital’s rule gives

\[ \lim_{x \to 0} \frac{6(3 \cos^2 3x - 3 \sin^2 3x)}{10} \]

which equals 1.8.

b. \( \lim_{x \to \infty} \sqrt{\frac{4x^3 - 2x}{9x^3 + 1}} \)

Since square roots are continuous, the value of this limit is the square root of the limit

\[ \lim_{x \to \infty} \frac{4x^3 - 2x}{9x^3 + 1} \]

which is \( \frac{4}{9} \). You can show that by various methods including the theorem that the limit as \( x \to \infty \) of a rational function whose numerator and denominator have the same degree is the quotient of the leading coefficients. Therefore, the answer is \( \sqrt{\frac{4}{9}} = \frac{2}{3} \).

c. Suppose that \( f'(x) = \sqrt{x^2 + 1} \). Use that information to evaluate

\[ \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \]

The limit is the definition of the derivative of \( f'(2) \), and that equals \( \sqrt{5} \).

3. [24; 6 points each part] Differentiation. Do not simplify your answers. Use parentheses properly.

a. For \( f(x) = x \ln x \), find \( f'(x) \).

Use the product rule.

\[ f'(x) = \ln x + x(1/x) = \ln x + 1. \]
b. Evaluate \( \frac{d}{dx} \tan^3(2x - \pi) \).

Use the chain rule twice. You’ll get

\[
3 \tan^2(2x - \pi) \sec^2(2x - \pi) 2.
\]

c. Let \( f(t) = \frac{e^t + t^2/3}{1 + \tan t} \). Find \( f'(t) \).

Use the quotient rule. You’ll find \( f'(t) \) equals

\[
\frac{(e^t + \frac{2}{3}t^{-1/3})(1 + \tan t) - (e^t + t^2/3) \sec^2 t}{(1 + \tan t)^2}.
\]

d. Let \( F(x) = \int_4^x t^4 + \ln(t^2 + 1) \, dt \). Find \( F'(x) \). (Hint: do not try to evaluate the integral.)

Use the version of the Fundamental Theorem of Calculus that tells you the derivative of the integral is the original function. Then

\[
F'(x) = \frac{x^5 + \ln(x^2 + 1)}{1 + \sqrt{x}}.
\]

4. [10] Determine the function \( f(x) \) whose derivative is \( f'(x) = 6x^2 - 4x + 2 \) and whose value at \( x = 1 \) is \( f(1) = 9 \).

Antidifferentiate \( f'(x) = 6x^2 - 4x + 2 \) to determine that \( f(x) = 2x^3 - 2x^2 + 2x + C \) for some value of \( C \). Since \( f(1) = 9 \), therefore \( 9 = 2 - 2 + 2 + C \), so \( C = 7 \). Therefore, \( f(x) = 2x^3 - 2x^2 + 2x + 7 \).

5. [10] A cylindrical aluminum can is to be constructed to have a volume of 1000 cubic cm. Let \( h \) denote the height of the can and \( r \) the radius of the base. Recall that the volume of a cylinder with height \( h \) and radius \( r \) of the base is \( V = \pi r^2 h \), and the total surface area is \( A = 2\pi r^2 + 2\pi rh \).

Determine the dimensions of the cylinder to minimize the surface area \( A \) of the can. Your final answer should indicate the values of \( r \) and \( h \).

Start with the equations

\[
1000 = V = \pi r^2 h, \quad A = 2\pi r^2 + 2\pi rh.
\]

Use the first to eliminate \( h \). \( h = \frac{1000}{\pi r^2} \), so the second equation becomes

\[
A = 2\pi r^2 + \frac{2000}{r}.
\]

Compute the derivative of \( A \) with respect to \( r \).

\[
\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}.
\]

Find the critical points by setting that derivative to 0 and solving the resulting equation for \( r \). That gives you

\[
r = \sqrt[3]{\frac{500}{\pi}}
\]

(which is about 5.42). That will minimize the surface area.

The value for \( h \) corresponding to that \( r \) is \( h = \frac{10000}{\pi(500/\pi)^{2/3}} \).

If you wanted to, you could use a little algebra shows you that \( h = 2r \), that is, the best shaped cylinder has the same height and diameter.

6. [12; 6 points each part] Evaluate the following integrals.

Note that the first one is an indefinite integral and the second one is a definite integral.

a. \( \int (5e^x + 3\cos x) \, dx \)

The general form of the antiderivative of the integrand is \( 5e^x + 3\sin x + C \).

b. \( \int_1^4 \left( x^2 + \frac{1}{2\sqrt{x}} \right) \, dx \)

An antiderivative is \( \frac{1}{3} x^3 + \sqrt{x} \), so the value of the definite integral is

\[
\left( \frac{1}{3} x^3 + \sqrt{x} \right)_{1}^{4} - \left( \frac{1}{3} x^3 + \sqrt{x} \right)_{1}^{3}.
\]

(which simplifies to 22).

7. [10] The graph of a function \( f(x) \) is drawn below. Its graph consists of three line segments.

Determine the value of the integral \( \int_{-1}^{4} f(x) \, dx \).

The region under the curve from \(-1\) to \(1\) is a triangle of area 2. The region under the curve from \(1\) to \(2\) is a rectangle of area 2. The region under the curve from \(2\) to \(\frac{10}{3}\) is \(\frac{1}{3}\), and the region above the curve from \(\frac{10}{3}\) to 4 is \(\frac{1}{3}\). So the value of the integral is \(2 + 2 + \frac{1}{3} - \frac{1}{3} = 5\).