Math 120 Calculus I
Final Answers
December 2016

Scale. [to be determined]

1. [22] Consider the function $f(x) = \dfrac{x}{1+x^2}$. Its derivative is $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$, and its second derivative is $f''(x) = \dfrac{4x^3-6x}{(1+x^2)^3}$.

a. [3] What are the x-intercepts and y-intercepts of f?

The only x-intercept is the origin and the y-intercept is also the origin. In other words, the graph of f only crosses either axis at the origin.

b. [3] What are the critical points for f?

The numerator $1-x^2$ of the derivative f' is 0 when $x = \pm 1$.

c. [3] What are the inflection points for f?

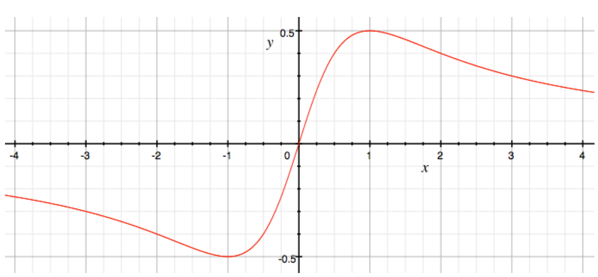
The numerator $4x^3 - 6x$ of the second derivative f'' is 0 when $x = 0, \pm\sqrt{3/2}$.

d. [4] Are there any vertical asymptotes? Are there any horizontal asymptotes?

There are no vertical asymptotes, but the x-axis is a horizontal asymptote since $\lim\limits_{x\to\pm\infty} f(x) = 0$.

e. [6] Sketch the graph of f. Show intercepts, critical points, inflection points, and asymptotes.

The curve looks like this, but it should be also annotated with the points mentioned above.



2. [15; 5 points each part] On limits. Evaluate each of the following limits in parts a and b if it exists, but if it doesn't then explain why.

a. $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{5x^2}$

One way to evaluate this limit is to rewrite it as

$$\lim_{x\to 0} \frac{\sin 3x}{3x} \frac{\sin 3x}{3x} \frac{9}{5}$$

which equals $\frac{9}{5}$ since $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

You could also use L'Hôpital's rule since this limit is of the indeterminant form 0/0. Then the limit is equal to

$$\lim_{x\to 0} \frac{6 \sin 3x \cos 3x}{10x}$$

It's still of the form 0/0, and another application of L'Hôpital's rule gives

$$\lim_{x\to 0} \frac{6(3\cos^2 3x - 3\sin^2 3x)}{10}$$

which equals 1.8.

b. $\lim\limits_{x\to\infty} \sqrt{\dfrac{4x^3-2x}{9x^3+1}}$

Since square roots are continuous, the value of this limit is the square root of the limit

$$\lim_{x\to\infty} \frac{4x^3-2x}{9x^3+1}$$

which is $\frac{9}{4}$. You can show that by various methods including the theorem that the limit as $x \to \infty$ of a rational function whose numerator and denominator have the same degree is the quotient of the leading coefficients. Therefore, the answer is $\sqrt{49} = 2/3$

c. Suppose that $f'(x) = \sqrt{x^2+1}$. Use that information to evaluate

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

The limit is the definition of the derivative of $f'(2)$, and that equals $\sqrt{5}$.

3. [24; 6 points each part] Differentiation. Do not simplify your answers. Use parentheses properly.

a. For $f(x) = x \ln x$, find $f'(x)$.

Use the product rule.

$$f'(x) = 1 \ln x + x(1/x) = \ln x + 1.$$

b. Evaluate $\dfrac{d}{dx}\tan^3(2x-\pi)$.

Use the chain rule twice. You'll get

$$3\tan^2(2x-\pi)\sec^2(2x-\pi)\,2.$$

c. Let $f(t)=\dfrac{e^t+t^{2/3}}{1+\tan t}$. Find $f'(t)$.

Use the quotient rule. You'll find $f'(t)$ equals

$$\frac{(e^t+\frac{2}{3}t^{-1/3})(1+\tan t)-(e^t+t^{2/3})\sec^2 t}{(1+\tan t)^2}$$

d. Let $F(x)=\displaystyle\int_4^x \frac{t^t+\ln(t^2+1)}{1+\sqrt{t}}\,dt$. Find $F'(x)$. (Hint: do not try to evaluate the integral.)

Use the version of the Fundamental Theorem of Calculus that tells you the derivative of the integral is the original function. Then

$$F'(x)=\frac{x^x+\ln(x^2+1)}{1+\sqrt{x}}.$$

4. [10] Determine the function $f(x)$ whose derivative is $f'(x)=6x^2-4x+2$ and whose value at $x=1$ is $f(1)=9$.

Antidifferentiate $f'(x)=6x^2-4x+2$ to determine that $f(x)=2x^3-2x^2+2x+C$ for some value of C. Since $f(1)=9$, therefore $9=2-2+2+C$, so $C=7$. Therefore, $f(x)=2x^3-2x^2+2x+7$.

5. [10] A cylindrical aluminum can is to be constructed to have a volume of 1000 cubic cm. Let h denote the height of the can and r the radius of the base. Recall that the volume of a cylinder with height h and radius r of the base is $V=\pi r^2h$, and the total surface area is $A=2\pi r^2+2\pi rh$. Determine the dimensions of the cylinder to minimize the surface area A of the can. Your final answer should indicate the values of r and h.

Start with the equations

$$1000=V=\pi r^2h,\text{ and }A=2\pi r^2+2\pi rh.$$

Use the first to eliminate h. $h=\dfrac{1000}{\pi r^2}$, so the second equation becomes

$$A=2\pi r^2+\frac{2000}{r}.$$

Compute the derivative of A with respect to r.

$$\frac{dA}{dr}=4\pi r-\frac{2000}{r^2}$$

Find the critical points by setting that derivative to 0 and solving the resulting equation for r. That gives you

$$r=\sqrt[3]{\frac{500}{\pi}}$$

(which is about 5.42). That will minimize the surface area. The value for h corresponding to that r is $h=\dfrac{10000}{\pi(500/\pi)^{2/3}}$.

If you wanted to, you could use a little algebra shows you that $h=2r$, that is, the best shaped cylinder has the same height and diameter.

6. [12; 6 points each part] Evaluate the following integrals. Note that the first one is an indefinite integral and the second one is a definite integral.

a. $\displaystyle\int(5e^x+3\cos x)\,dx$

The general form of the antiderivative of the integrand is $5e^x+3\sin x+C$.

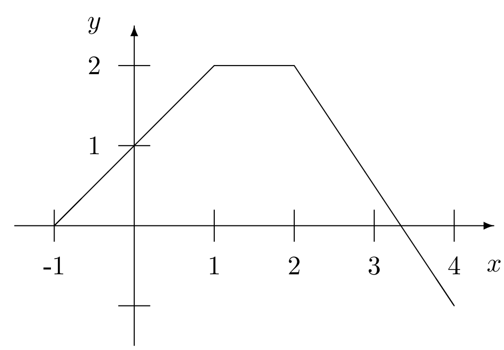
b. $\displaystyle\int_1^4\left(x^2+\frac{1}{2\sqrt{x}}\right)dx$

An antiderivative is $\frac{1}{3}x^3+\sqrt{x}$, so the value of the definite integral is

$$(\tfrac{1}{3}x\,4^3+\sqrt{4})-(\tfrac{1}{3}x\,1^3+\sqrt{1})$$

(which simplifies to 22).

7. [10] The graph of a function $f(x)$ is drawn below. Its graph consists of three line segments.



Determine the value of the integral $\displaystyle\int_{-1}^4 f(x)\,dx$.

The region under the curve from -1 to 1 is a triangle of area 2. The region under the curve from 1 to 2 is a rectangle of area 2. The region under the curve from 2 to $\frac{10}{3}$ is $\frac{4}{3}$, and the region above the curve from $\frac{10}{3}$ to 4 is $\frac{1}{3}$. So the value of the integral is $2+2+\frac{4}{3}-\frac{1}{3}=5$.