Math 120 Calculus I

Second Test Answers
November 2016

Scale. 90–104 A. 79–89 B. 60–78 C. 40–59 D. Median 94.

1. [10] On implicit differentiation. The point $(0,1)$ lies in the curve $2x^2 = y^2(x+1) - 1$. Determine the slope of the line tangent to the curve at that point.

Use implicit differentiation and the product rule on the equation $2x^2 = y^2(x+1) - 1$ to get

$$4x = 2y(x+1)\frac{dy}{dx} + y^2$$

Solve for $\dfrac{dy}{dx}$ to find $\frac{dy}{dx} = \frac{4x-y^2}{2y(x+1)}$. Then the slope of the line tangent to the curve at $(0,1)$ is that derivative when $x = 0$ and $y = 1$, namely $-\frac{1}{2}$.

2. [10; 5 points each part] Graphs and derivatives.

a. Sketch the graph $y = f(x)$ of a function for which is continuous everywhere and differentiable everywhere except at $x = 4$.

Your graph should be a smooth continuous curve except at $x = 4$ where there should be a corner or cusp in the curve.

b. Sketch the graph $y = f(x)$ of a function that is differentiable everywhere and whose derivative is 0 at $x = 3$ and $x = 5$ but the derivative is nonzero everywhere else.

Your graph should be a smooth curve where the only horizontal tangents occur at $x = 3$ and $x = 5$. One way to do that is to have a maximum at 3 and a minimum at 5 or vice versa, but there are other ways besides those.

3. [15] Recall the definition of derivatives in terms of limits, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. Use that definition to show that the derivative of $f(x) = 5 + \dfrac{1}{x}$ is $f'(x) = -\dfrac{1}{x^2}$. (Do not use any of the rules of differentiation, just the definition.)

Your details may be slightly different, but should follow more or less the following steps:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(5 + \frac{1}{x+h}) - (5 + \frac{1}{x})}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{(x+h)x}}{h} \\
&= \lim_{h\to 0} \frac{-1}{(x+h)x} \\
&= \frac{-1}{x^2}
\end{aligned}$$

4. [12] On logarithmic differentiation. The function $y = f(x) = x^x$ cannot be differentiated by the power rule since the exponent is not constant, and it can't be differentiated by the exponential rule since the base is not constant, but you can find its derivative with logarithmic differentiation. Find its derivative. Show your work, and write carefully. Express your answer $f'(x)$ in terms of x.

First take the log of the equation $y = x^x$ to get

$$\ln y = x \ln x.$$

Then differentiate with respect to x to get

$$\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$$

Then multiply by y to find the derivative of y.

$$y' = y(\ln x + 1) = x^x(\ln x + 1).$$

5. [45; 9 points each part] Differentiate the following functions. Do not simplify your answers. Use parentheses properly.

a. $f(t) = 7t^4 - 4t + 6$

You only need the power rule for this. $f'(x) = 28t^3 - 4$.

b. $g(x) = (3x^2 - 5)(\sin x)$

Use the product rule. $g'(x) = 6x \sin x + (3x^2 - 5)\cos x$.

c. $y = \dfrac{e^x}{1 + x^2}$

The quotient rule is needed here.

$$y' = \frac{e^x(1 + x^2) - e^x(2x)}{(1 + x^2)^2}$$

d. $f(x) = x \arctan x$. (Note that the inverse tangent function $\arctan x$ is often written $\tan^{-1} x$, but it does not equal $(\tan x)^{-1}$.)

Use the product rule and the fact that $(\arctan x)' = \dfrac{1}{1+x^2}$.

$$f'(x) = \arctan x + \frac{x}{1+x^2}$$

e. $f(x) = \sqrt{\ln x}$

Use the chain rule

$$f'(x) = \frac{1}{2\sqrt{\ln x}}(\ln x)' = \frac{1}{2x\sqrt{\ln x}}$$

6. [12] On related rates.

The formula for the volume of a circular cylinder is $V = \pi r^2 h$ where h is its height and r is the radius of the base.

Suppose that the height is constantly 10 inches, but the radius is increasing at the rate of 1 inch per minute. Determine the rate of change of the volume when the radius is 6 inches.

Differentiate the equation $V = \pi r^2 h$ with respect to time t keeping in mind that h is constant.

$$\frac{dV}{dt} = \pi 2r \frac{dr}{dt} h$$

When $\dfrac{dr}{dt} = 1$ and $r = 6$, this rate is equal to $\dfrac{dV}{dt} = 120\pi$ inches squared per minute.