This is a small note to show that the number \( e \) is equal to a limit, specifically
\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.
\]
Sometimes this is taken to be the definition of \( e \), but I’ll take \( e \) to be the base of the natural logarithms.

For a positive number \( x \) the natural logarithm of \( x \) is defined as the integral
\[
\ln x = \int_1^x \frac{1}{t} \, dt.
\]
Then \( e \) is the unique number such that \( \ln e = 1 \), that is,
\[
1 = \int_1^e \frac{1}{t} \, dt.
\]
The natural exponential function \( e^x \) is the function inverse to \( \ln x \), and all the usual properties of logarithms and exponential functions follow.

Here’s a synthetic proof that \( e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \).

A synthetic proof is one that begins with statements that are already proved and progresses one step at a time until the goal is achieved. A defect of synthetic proofs is that they don’t explain why any step is made.

**Proof.** Let \( t \) be any number in an interval \([1, 1 + \frac{1}{n}]\). Then
\[
\frac{1}{1 + \frac{1}{n}} \leq \frac{1}{t} \leq 1.
\]
Therefore
\[
\int_1^{1 + \frac{1}{n}} \frac{1}{1 + \frac{1}{n}} \, dt \leq \int_1^{1 + \frac{1}{n}} \frac{1}{t} \, dt \leq \int_1^{1 + \frac{1}{n}} 1 \, dt.
\]
The first integral equals \( \frac{1}{n+1} \), the second equals \( \ln(1 + \frac{1}{n}) \), and the third equals \( \frac{1}{n} \). Therefore,
\[
\frac{1}{n+1} \leq \ln \left(1 + \frac{1}{n}\right) \leq \frac{1}{n}.
\]
Exponentiating, we find that
\[
e^{\frac{1}{n+1}} \leq 1 + \frac{1}{n} \leq e^{\frac{1}{n}}.
\]
Taking the \((n + 1)\)st power of the left inequality gives us
\[
e \leq \left(1 + \frac{1}{n}\right)^{n+1}
\]
while taking the \(n\)th power of the right inequality gives us
\[
\left(1 + \frac{1}{n}\right)^n \leq e.
\]
Together, they give us these important bounds on the value of \( e \):
\[
\left(1 + \frac{1}{n}\right)^n \leq e \leq \left(1 + \frac{1}{n}\right)^{n+1}.
\]
Divide the right inequality by \( 1 + \frac{1}{n} \) to get
\[
\frac{e}{1 + \frac{1}{n}} \leq \left(1 + \frac{1}{n}\right)^n
\]
which we combine with the left inequality to get
\[
\frac{e}{1 + \frac{1}{n}} \leq \left(1 + \frac{1}{n}\right)^n \leq e.
\]
But both \( \frac{e}{1 + \frac{1}{n}} \to e \) and \( e \to e \), so by the pinching theorem \( \left(1 + \frac{1}{n}\right)^n \to e \), also.

Q.E.D.