Fermat computes an integral
Math 121 Calculus II
Spring 2015

Pierre de Fermat (1601–1665), along with Descartes (1596–1650), invented the xy-coordinate system and analytic geometry. Besides developing analytic geometry, Fermat and Descartes were also early researchers in the subject that we now call calculus.

Here's how Fermat evaluated the area under the graph of a power function $y = x^n$, that is, how he determined what we now write as

$$\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1}.$$

Fermat wasn't the only one to find this integral. Bonaventura Cavalieri (1598–1647) published that result for positive integers n in 1647 using using a form of infinitesimal analysis he had used since 1626. Fermat's proof, however, is easier to follow, and it applies to positive rational numbers n as well.

His general method is the same as those before and after him. He approached the area by rectangular estimates.

Fermat partitioned the interval $[0, a]$ is a clever way. Not all the subintervals were the same size, but he had shorter intervals near 0 and longer intervals near a. Now, that seems like it wouldn't be as good an approximation as making all the subintervals the same length, but it's good enough, and it made the calculations work out. In fact, he selected this partition so that the sum of the area of rectangles above the partition would be an infinite geometric series.

Geometric series. We haven't looked at infinite geometric series yet in our course, but they were already understood over two centuries before Fermat.

We'll study them in detail later, but we only need to know a little bit about them now. An infinite geometric series is an infinite sum of terms where the ratio of each term to the next is a constant, usually denoted r. If the first term of a series is denoted a, and the ratio lies between -1 and 1, then the sum of the series is $\dfrac{a}{1-r}$, something Fermat knew well.

Fermat's upper rectangular estimates. Choose any positive number E less than 1, and partition the interval $[0, a]$ so the last subinterval is $[Ea, a]$, the one before that $[E^2a, Ea]$, then $[E^3a, E^2a]$, etc. There are infinitely many subintervals. The k^{th} subinterval from the right is $[E^{k+1}a, E^ka]$, and its length is $E^ka - E^{k+1}a$.

Figure 1 displays Fermat's rectangles for $y = x^{3/2}$.

On each subinterval $[E^{k+1}a, E^ka]$, put a rectangle of shortest height that encloses the area under the curve $y = x^n$ above that subinterval. That height occurs at the right endpoint E^ka, so that height is $(E^ka)^n$. Then that rectangle will have area $(E^ka)^n(E^ka - E^{k+1}a)$, which simplifies as

$$E^{(n+1)k}a^{n+1}(1 - E).$$

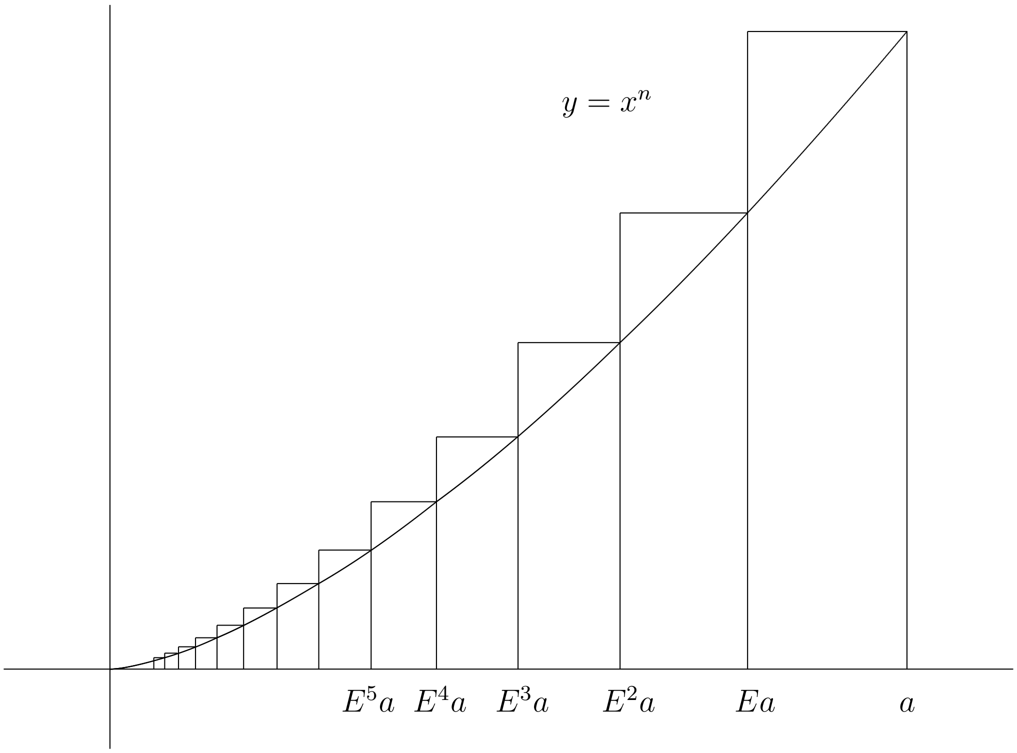

Figure 1: Fermat's rectangles for $y = x^n$

Fermat then computed the total area of all the rectangles, which is

$$\sum_{k=0}^{\infty} E^{(n+1)k} a^{n+1}(1-E).$$

Thats a geometric series whose first term is $a^{n+1}(1-E)$ and whose ratio is E^{n+1}, so its sum is equal to

$$\frac{a^{n+1}(1-E)}{1-E^{n+1}} = a^{n+1}\,\frac{1-E}{1-E^{n+1}}.$$

That's an upper rectangular estimate for the area under the curve. As the partition becomes finer, that is, as E approaches 1, this upper rectangular estimate will approach the area under the curve. That means we need to see what $\dfrac{1-E}{1-E^{n+1}}$ approaches as $E \to 1$.

We can use l'Hôptal's rule to evaluate that limit, but it hadn't been invented in Fermat's time. We'll look at how Fermat evaluated that limit when n is a positive integer. Note that $\dfrac{1-E}{1-E^{n+1}}$ can be rewritten as $\dfrac{1}{1+E+E^2+\cdots+E^n}$, so as $E \to 1$, $\dfrac{1}{1+E+E^2+\cdots+E^n} \to \dfrac{1}{n+1}$. Fermat also showed that when n was a positive rational number p/q that the limit was also $\dfrac{1}{n+1}$. (An interesting exercise. See if you can do it.)

Therefore, the upper rectangular estimate approaches $\dfrac{a^{n+1}}{n+1}$.

A similar analysis shows that lower rectangular estimates also approaches this same number. Since the area under the curve lies between the lower and upper rectangular estimates, and both approach this same number, therefore the area under the curve does equal $\dfrac{a^{n+1}}{n+1}$.

Source: Carl B. Boyer and Uta C. Merzbach's *A History of Mathematics*, second edition, New York, Wiley & Sons, 1989, pages 350–352.

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