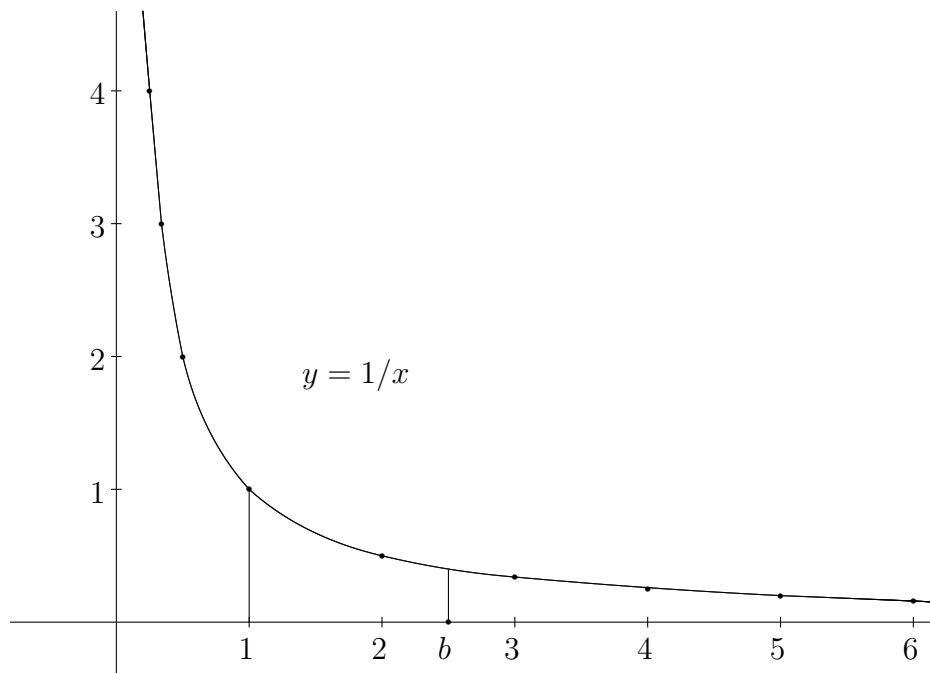


Logarithms  
 Math 121 Calculus II  
 D Joyce, Spring 2015

You know all about logarithms already, but one of the best ways to define and prove properties about them is by means of calculus. We'll do that here. The main reason, however, for going on this excursion is to see how logic is used in formal mathematics. We'll use what we know about calculus to prove statements about logs and exponents.

**A definition in terms of areas.** Consider the area below the standard hyperbola  $y = 1/x$ , above the  $x$ -axis, and between the vertical lines  $x = 1$  and  $x = b$  where  $b$  is a positive number. We'll treat this as a signed area, so that when  $b > 1$  the area is counted positively, and when  $0 < b < 1$  we'll count it negatively. In other words, consider the integral  $\int_1^b \frac{1}{x} dx$ .



**Definition 1.** The *natural logarithm*, or more simply the *logarithm*, of a positive number  $b$ , denoted  $\ln b$  or  $\log b$  is defined as

$$\ln b = \int_1^b \frac{1}{x} dx.$$

The notation  $\log b$  is standard in mathematics, but I'll use  $\ln b$  since that's the notation in our textbook.

## Properties of the logarithm function.

**Theorem 2.**  $\ln 1 = 0$

*Proof.* Since an integral whose lower limit of integration equals its upper limit of integration is 0, therefore

$$\ln 1 = \int_1^1 \frac{1}{x} dx = 0.$$

Q.E.D.

**Theorem 3.** The function  $\ln x$  is differentiable and continuous on its domain  $(0, \infty)$ , and its derivative is  $\frac{d}{dx} \ln x = \frac{1}{x}$ .

*Proof.* By the inverse of the Fundamental Theorem of Calculus, since  $\ln x$  is defined as an integral, it is differentiable and its derivative is the integrand  $1/x$ . As every differentiable function is continuous, therefore  $\ln x$  is continuous. Q.E.D.

**Theorem 4.** These two properties,  $\ln 1 = 0$  and  $\frac{d}{dx} \ln x = \frac{1}{x}$ , characterize the logarithm.

*Proof.* Any function  $f(x)$  whose derivative is  $f'(x) = 1/x$  differs from  $\ln x$  by a constant, so if it agrees with  $\ln x$  for one value of  $x$ , namely  $x = 1$ , then that constant is 0, so  $f(x) = \ln x$ . Q.E.D.

**Theorem 5.** The logarithm of a product of two positive numbers is the sum of their logarithms, that is,  $\ln xy = \ln x + \ln y$ .

*Proof.* We'll use a general principle here that if two functions have the same derivative on an interval and they agree for one particular argument, then they are equal. It's a useful principle that can be used to prove identities like this.

Treat the left hand side of the equation as a function of  $x$  leaving  $y$  as a constant, thus,  $f(x) = \ln xy$ . Likewise, let the right hand side of the equation be  $g(x) = \ln x + \ln y$  where again  $y$  is a constant and  $x$  is a variable.

Then, by the chain rule for derivatives,

$$\frac{d}{dx} f(x) = \frac{d}{dx} (\ln xy) = \frac{1}{xy} \frac{d}{dx} xy = \frac{y}{xy} = \frac{1}{x}.$$

We also have

$$\frac{d}{dx} g(x) = \frac{d}{dx} (\ln x + \ln y) = \frac{1}{x} + 0 = \frac{1}{x}.$$

Since  $f$  and  $g$  have the same derivatives on the interval  $(0, \infty)$ , therefore they differ by a constant. But taking  $x = 1$ ,  $f(1) = \ln y$  and  $g(1) = \ln 1 + \ln y = \ln y$ , so the constant they differ by is 0, that is to say,  $f = g$ . Q.E.D.

**Theorem 6.** The logarithm of a quotient of two positive numbers is the difference of their logarithms, that is,  $\ln x/y = \ln x - \ln y$ .

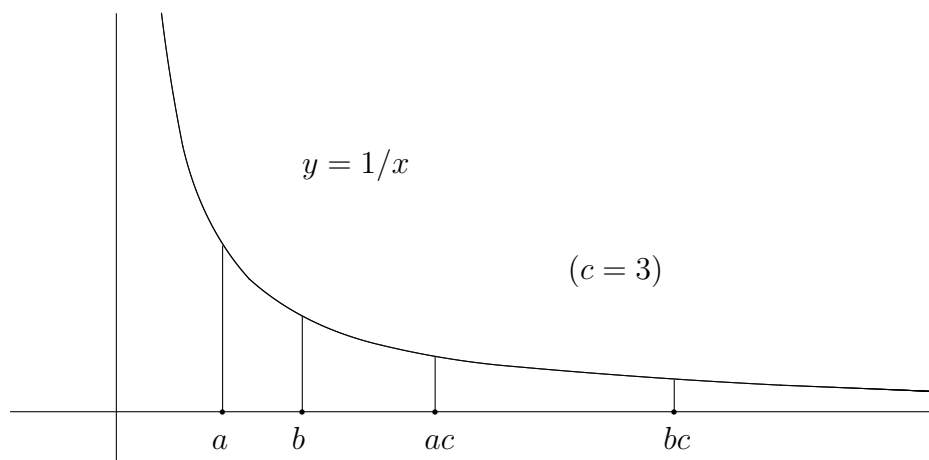
*Proof.* Although the same kind of proof could be given as in the preceding theorem, we can also derive this from the preceding theorem. Let  $z = x/y$  so that  $x = yz$ . Since  $\ln yz = \ln y + \ln z$ , therefore  $\ln yz - \ln y = \ln z$ , that is,  $\ln x - \ln y = \ln x/y$ . Q.E.D.

**Theorem 7.** The logarithm of the reciprocal of a positive number is the negation of the logarithm of that number, that is,  $\ln 1/y = -\ln y$ .

*Proof.* Using the preceding theorem,  $\ln 1/y = \ln 1 - \ln y = 0 - \ln y = -\ln y$ . Q.E.D.

These theorems can be proved in a more geometric manner using properties of transformations of area.

The graph of the hyperbola  $y = 1/x$  has a special property. If you compress the plane vertically by a factor of  $c$ , then expand the plane horizontally by that same factor, then the hyperbola falls on itself. Start with the point  $(x, 1/x)$ , compress vertically to get  $(x, 1/cx)$ , then expand horizontally to get  $(cx, 1/cx)$ , another point on the graph.



The integral  $\int_a^b \frac{1}{x} dx$  describes the area  $A$  of the region under the hyperbola above the interval  $[a, b]$ . That area was studied by Gregory of Saint-Vincent in 1647. He proved that the area above the interval  $[a, b]$  was the same as the area above the interval  $[ca, cb]$  for any positive constant  $c$ . One argument that he gave used rectangular approximations.

When the plane is compressed vertically by a factor of  $c$ , that region is compressed into a region of area  $A/c$ . When it's expanded horizontally, the resulting region expands back to an area  $A$ , but that region is the area under the hyperbola above the interval  $[ca, cb]$  which has area  $\int_{ca}^{cb} \frac{1}{x} dx$ . Thus

$$\int_a^b \frac{1}{x} dx = \int_{ca}^{cb} \frac{1}{x} dx.$$

That translates into the following identity for logarithms

$$\ln b - \ln a = \ln cb - \ln ca.$$

Setting  $a = 1$ ,  $b = x$ , and  $c = y$  yields the identity  $\ln xy = \ln x + \ln y$ , while setting  $a = y$ ,  $b = x$ , and  $c = 1/y$  yields the identity  $\ln x/y = \ln x - \ln y$ .

**Theorem 8.** If  $n$  is an integer and  $x$  a positive number then  $\ln x^n = n \ln x$ .

*Proof.* First, consider the case when  $n = 0$ . Then  $\ln x^n = \ln 1 = 0 = 0 \ln x = n \ln x$ .

Next, consider the case when  $n$  is a positive integer. Then  $n$  is the sum of  $n$  1's,  $n = 1 + 1 + \cdots + 1$ . Therefore,  $x^n$  is the product of  $n$   $x$ 's,  $x^n = x \cdot x \cdots x$ , so

$$\ln x^n = \ln(x \cdot x \cdots x) = \ln x + \ln x + \cdots + \ln x = n \ln x.$$

Finally, consider the case when  $n$  is a negative integer. Then  $\ln x^n = \ln((1/x)^{-n})$ , and since  $-n$  is a positive integer, we have by the previous case that  $\ln((1/x)^{-n}) = -n \ln(1/x)$ , which equals  $n \ln x$ . Q.E.D.

**Theorem 9.** If  $n$  is an integer and  $x$  a positive number then  $\ln \sqrt[n]{x} = \frac{1}{n} \ln x$ .

*Proof.* Since  $n \ln \sqrt[n]{x} = \ln((\sqrt[n]{x})^n) = \ln x$ , divide by  $n$  to get the desired identity. Q.E.D.

**Theorem 10.** If  $y$  is a rational number and  $x$  a positive number then  $\ln x^y = y \ln x$ .

*Proof.* Let  $y$  be the rational number  $m/n$  with  $n$  positive. Then

$$\ln x^y = \ln x^{m/n} = \ln((\sqrt[n]{x})^m) = m \ln \sqrt[n]{x} = \frac{m}{n} \ln x = y \ln x.$$

Q.E.D.

**Theorem 11.** The function  $\ln x$  is an increasing one-to-one function on its domain  $(0, \infty)$ .

*Proof.* Since its derivative  $1/x$  is positive, therefore it's increasing. Every increasing function on an interval is one-to-one. Q.E.D.

**Theorem 12.** The graph  $y = \ln x$  of the function  $\ln x$  is concave downward.

*Proof.* The second derivative of  $\ln x$  is  $-1/x^2$  which is negative, therefore its graph is concave downward. Q.E.D.

**Theorem 13.** The range of the function  $\ln x$  includes all real numbers.

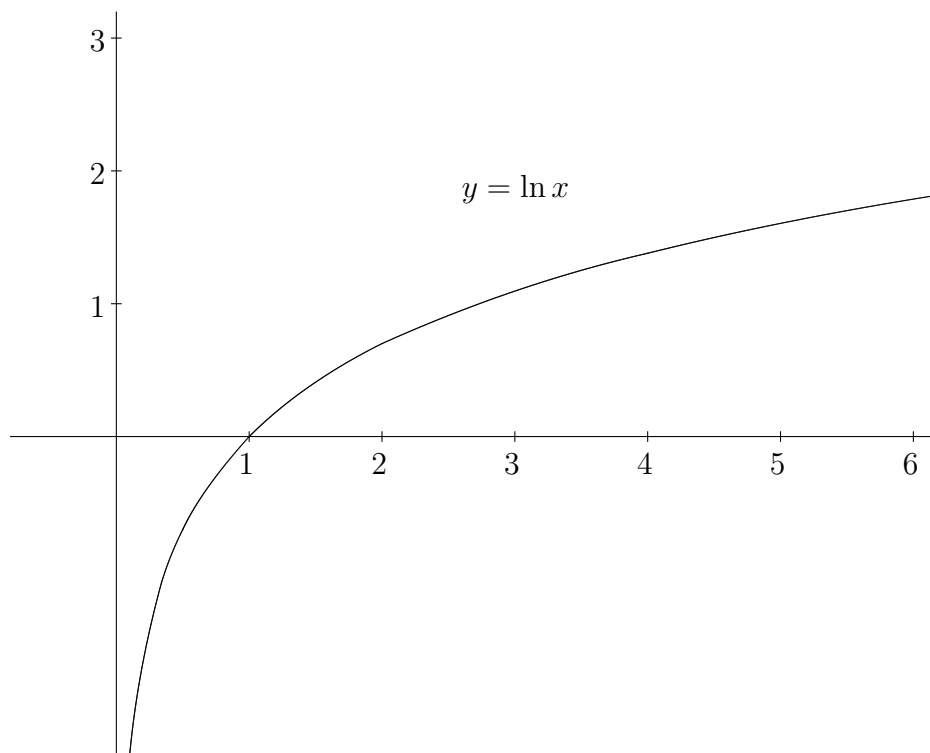
*Proof.* Let  $b$  be any number greater than 1, then  $c = \ln b > 0$ . Then multiples  $nc$  approach  $\infty$  as  $n$  increases to  $\infty$ . But  $nc = n \ln b = \ln b^n$ , so the values of the function  $\ln x$  grow arbitrarily large. Also,  $-nc = \ln b^{-n}$  approaches  $-\infty$ .

Since the function  $\ln x$  is a continuous function, it takes on all intermediate values as well. Therefore its range includes all real numbers. Q.E.D.

**Theorem 14.**  $\lim_{x \rightarrow \infty} \ln x = \infty$ , and  $\lim_{x \rightarrow 0^+} \ln x = -\infty$ .

*Proof.* Those limits hold since the function  $\ln x$  is an increasing function with domain  $(0, \infty)$  whose range includes all real numbers. Q.E.D.

We now have enough qualitative information to sketch a graph of  $y = \ln x$ .



### The number $e$ .

**Definition 15.** As the function  $\ln x$  is a one-to-one function with domain  $(0, \infty)$  and range  $(-\infty, \infty)$ , there is exactly one number whose logarithm equals 1, it is denoted  $e$ . Thus  $\ln e = 1$ .

In other words,  $e$  is the number such that the area equals 1 under the hyperbola  $y = 1/x$  and above the interval  $[1, e]$ .

We can estimate the value of  $e$  from its definition.

**Theorem 16.**  $\ln 2$  is less than 1, while  $\ln 3$  is greater than 1. Therefore,  $e$  lies between 2 and 3.

*Proof.* To show that  $\ln 2$  is less than 1, note that the region under the hyperbola  $y = 1/x$  over the interval  $[1, 2]$  lies inside a square on that interval. Since that square has area 1, that region, whose area is  $\ln 2$  is less than 1.

To show that  $\ln 3$  is greater than 1, use a lower rectangular estimate of the area under the hyperbola over  $[1, 3]$  using a uniform partition into 8 parts. The six rectangles have heights  $\frac{4}{5}, \frac{4}{6}, \frac{4}{7}, \dots, \frac{4}{12}$ , and each has a width of  $\frac{1}{4}$ . The lower estimate is, therefore,  $\frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{12}$ , which is about 1.127, greater than 1.

Since  $\ln 2 < 1 < \ln 3$ , and  $\ln x$  is an increasing function, therefore  $e$ , the number whose logarithm is 1, lies between 2 and 3. Q.E.D.

The definition is not the fastest way to approximate  $e$ . One property of  $e$  that quickly estimates  $e$  is an expression for  $e$  as the infinite sum (also called a *series*)

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots .$$

(Here,  $n!$  is the product of the integers from 1 through  $n$ .) We won't prove that here. You can find good approximations of  $e$  by truncating this infinite sum to a finite sum. For example,  $2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.71806$  which is correct to 3 decimal places.

Another property of  $e$  we won't prove here, but is important in many applications is that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

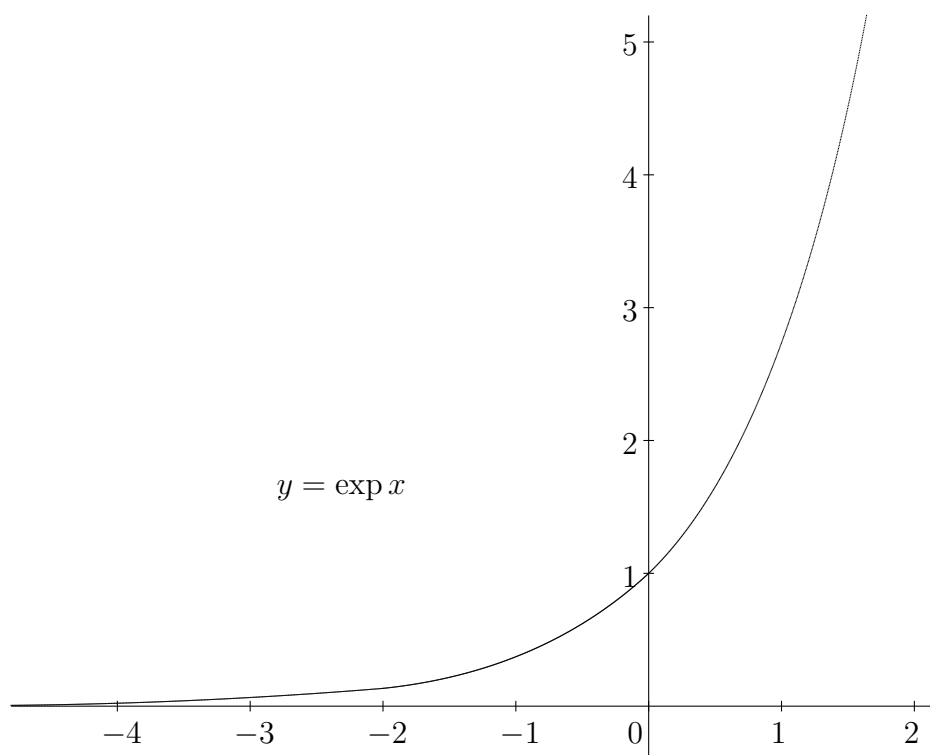
We'll prove that later.

### The exponential function and its properties.

**Definition 17.** The *exponential function*  $\exp x$  is the function inverse to the logarithm function  $\ln x$ :

$$y = \exp x \quad \text{if and only if} \quad x = \ln y.$$

It's domain includes all real numbers, and its range is the interval  $(0, \infty)$ .



**Theorem 18.** For each positive number  $x$ ,  $\exp(\ln x) = x$ , and for each number  $x$ ,  $\ln(\exp x) = x$ . In particular,  $\exp 0 = 1$ , and  $\exp 1 = e$ .

*Proof.* The first two identities follow directly from the definition, and the last two are particular instances of the first when  $x = 1$  and  $x = e$ , respectively. Q.E.D.

Later, after a few theorems, we'll define for any positive number  $a$  the exponential  $a^x$  as  $\exp(x \ln a)$ , so, in particular,  $e^x = \exp(x \ln e) = \exp x$ . After that we'll generally use the notation  $e^x$  in preference to  $\exp x$ .

**Theorem 19.** The exponential function  $\exp x$  is differentiable and continuous. Its derivative is itself,  $\frac{d}{dx} \exp x = \exp x$ . It is an increasing function, and its graph  $y = \exp x$  is concave upward everywhere.

*Proof.* Begin with the identity  $x = \ln \exp x$ . Take the derivative with respect to  $x$  to conclude by the chain rule that  $1 = \frac{1}{\exp x} \frac{d}{dx} \exp x$ . Therefore  $\exp x = \frac{d}{dx} \exp x$ .

Since its derivative, which  $\exp x$ , is positive,  $\exp x$  is an increasing function. Since its second derivative, which is also  $\exp x$ , is positive, its graph is concave upward. Q.E.D.

**Theorem 20.** The exponential of a sum is the product of the exponentials, that is,  $\exp(a + b) = \exp a \exp b$ .

*Proof.* Let  $c = \exp a$  and  $d = \exp b$ . Since  $\ln cd = \ln c + \ln d$ , therefore  $\exp \ln cd = \exp(\ln c + \ln d)$ . But  $cd = \exp \ln cd$ , so  $cd = \exp(\ln c + \ln d)$ . Also,  $a = \ln c$  and  $b = \ln d$ . Thus  $\exp a \exp b = \exp(a + b)$ . Q.E.D.

**Theorem 21.** The exponential of a difference is the quotient of the exponentials, that is,  $\exp(a - b) = \frac{\exp a}{\exp b}$ .

*Proof.* Substitute  $a - b$  for  $a$  in the identity  $\exp(a + b) = \exp a \exp b$  to get  $\exp a = \exp(a - b) \exp b$ . Then divide by  $\exp b$  to get the desired identity. Q.E.D.

**Theorem 22.** The exponential of a negation is the reciprocal of the exponential, that is,  $\exp(-b) = \frac{1}{\exp b}$ .

*Proof.* Set  $a = 0$  in the preceding identity. Q.E.D.

**General exponentiation.** Up until now, exponentiation has only been defined when the exponent is a rational number. If  $y = m/n$ , then  $x^y$  means  $(\sqrt[n]{x})^m$ . We'll extend exponentiation to irrational exponents by defining, for  $x$  positive,  $x^y = \exp(y \ln x)$ . But before we can extend the definition that way, we have to prove that this agrees with exponentiation when  $y$  is a rational number, that is, we need to prove that  $x^y = \exp(y \ln x)$  when  $x$  is positive and  $y$  is a rational number. The next few theorems lead to that result.

**Theorem 23.** For any integer  $n$  and any number  $x$ ,  $\exp nx = (\exp x)^n$ .

*Proof.* In the case when  $n = 0$ ,  $\exp 0x = \exp 0 = 1 = 1^0 = (\exp x)^0$ .

In the case when  $n$  is positive, it is the sum of  $n$  1s,  $n = 1 + 1 + \cdots + 1$ . So  $\exp nx = \exp(1 + 1 + \cdots + 1)x = \exp(x + x + \cdots + x) = \exp x \exp x \cdots \exp x = (\exp x)^n$ .

In the case when  $n$  is negative,  $\exp nx = \exp(-(-n)x) = 1/\exp(-n)x$ , which, by the preceding case, equals  $1/(x^{-n}) = x^n$ . Q.E.D.

**Theorem 24.** For any positive integer  $n$  and any number  $x$ ,  $\exp x/n = \sqrt[n]{\exp x}$ .

*Proof.* By the preceding theorem  $\left(\exp \frac{x}{n}\right)^n = \exp n \left(\frac{x}{n}\right) = \exp x$ . Taking  $n^{\text{th}}$  roots we get the desired identity. Q.E.D.

**Theorem 25.** For any rational number  $y$  and any number  $x$ ,  $\exp xy = (\exp x)^y$ .

*Proof.* Let  $y$  be the rational number  $m/n$  with  $n$  positive. Then

$$\exp xy = \exp \frac{mx}{n} = \left( \exp \frac{x}{n} \right)^m = (\sqrt[n]{x})^m = x^{m/n} = x^y.$$

Q.E.D.

**Theorem 26.** When  $x$  is positive and  $y$  is a rational number,  $x^y = \exp(y \ln x)$ .

*Proof.* Let  $a = \ln x$  so that  $x = \exp a$ . Then, by the previous theorem,  $x^y = (\exp a)^y = \exp ay = \exp(y \ln x)$ . Q.E.D.

Now that we have that theorem, we can extend exponentiation with arbitrary powers.

**Definition 27.** Exponentiation for a positive base  $b$  and any power  $a$  is defined by

$$b^a = \exp(a \ln b).$$

**Theorem 28.**  $\exp x = e^x$

*Proof.*  $e^x = \exp(x \ln e) = \exp x$ .

Q.E.D.

After this, we'll generally use the notation  $e^x$  rather than  $\exp x$ , except when the exponent  $x$  is a complicated expression that would be difficult to read as a small exponent.

All the usual properties of powers hold when the exponents are irrational, and their proofs follow directly from the definition and the preceding theorems. We'll leave out their proofs. In each of these identities, it is assumed that the bases of exponentiation are all positive numbers.

$$\begin{array}{llll} x^0 = 1 & & x^1 = x & & 1^x = 1 \\ & & x^{-1} = \frac{1}{x} & & x^{1/n} = \sqrt[n]{x} \\ x^{y+z} = x^y x^z & & x^{y-z} = \frac{x^y}{x^z} & & x^{-z} = \frac{1}{x^z} \\ & & (xy)^z = x^z y^z & & (x^y)^z = x^{yz} \\ x^y = e^{y \ln x} & & & & \ln x^y = y \ln x \end{array}$$

There are a few properties of general exponentiation that should be proved as they relate to differentiation and integration.

First, we'll state and prove the general power rules for differentiation and integration.

**Theorem 29.** The power rule for differentiation  $\frac{d}{dx} x^a = ax^{a-1}$  holds for all numbers  $a$ .

*Proof.*  $\frac{d}{dx} x^a = \frac{d}{dx} \exp(a \ln x) = \exp(a \ln x) \frac{d}{dx} (a \ln x) = x^a \frac{a}{x} = ax^{a-1}$  Q.E.D.

**Theorem 30.** The power rule for integration holds when  $a \neq -1$

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C.$$



*Proof.* By the preceding theorem, the derivative of  $x^{a+1}/(a+1)$  equals the integrand  $x^a$ .  
Q.E.D.

Next, we'll state and prove the general exponential rules for differentiation and integration. Note that you use the power rules when the powers are constants, but you use the exponential rules when the bases are constant. When both the base and exponent are variable, rewriting  $b^a$  as  $e^{a \ln b}$  should work. Also, the method of logarithmic differentiation will work in finding derivatives of such functions.

**Theorem 31.** The exponential rule for differentiation is

$$\frac{d}{dx} a^x = a^x \ln a.$$

*Proof.*  $\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a.$  Q.E.D.

**Theorem 32.** The exponential rule for integration is

$$\int a^x dx = \frac{a^x}{\ln a} + C.$$

*Proof.* By the preceding theorem, the derivative of  $\frac{a^x}{\ln a}$  equals the integrand  $a^x$ . Q.E.D.

**General logarithms.** When calculus is involved, natural logarithms are usually used. For special purposes, other logarithms are used, mainly logarithms with base 10 or with base 2.

**Definition 33.** If  $b$  is a positive number other than 1, then the *logarithm with base  $b$*  is defined by

$$\log_b x = \frac{\ln x}{\ln b}.$$

**Theorem 34.**  $\log_e x = \ln x.$

*Proof.*  $\log_e x = \frac{\ln x}{\ln e} = \ln x.$  Q.E.D.

**Theorem 35.** The function  $\log_b x$  has domain  $(0, \infty)$  and range all numbers. It is inverse to the exponential function  $b^x$ .

*Proof.* As it's just scaled by a factor of  $\ln b$ , it will have the same domain, and its range will still be all real numbers.

It's inverse to the exponential function since  $y = \log_b x$  means  $y = \frac{\ln x}{\ln b}$ , which is equivalent to  $y \ln b = \ln x$ . Since the function  $\ln x$  is inverse to the function  $\exp x$ , that last condition is equivalent to  $x = \exp(y \ln b)$ , but that says  $x = b^y$ . Thus, the function  $\log_b x$  is inverse to the function  $b^x$ .  
Q.E.D.

All the usual properties of general logarithms follow from the definition and previous theorems. In particular, the change of base formula holds,  $\log_c a = \frac{\log_b a}{\log_b c}$ .

The derivative of the function  $\log_b x$  is easy to compute.

**Theorem 36.**  $\frac{d}{dx} \log_b x = \frac{1}{x \ln b} = \frac{\log_b e}{x}$ .

*Proof.*  $\frac{d}{dx} \log_b x = \frac{d}{dx} \frac{\ln x}{\ln b} = \frac{1}{x \ln b}$ . A special case of the change of base formula above, namely  $a = b$  shows that  $\log_c a = \frac{1}{\log_a c}$ , which implies the second equality in the statement of the theorem. Q.E.D.

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